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Bifurcation for an elliptic problem with nonlinear boundary conditions

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Abstract. This paper gives a survey over bifurcation problems for elliptic equations with nonlinear boundary conditions depending on a real parameter. We consider an elliptic equation with a nonlinear boundary condition which is asymptotically linear at infinity and which depends on a parameter. As the parameter crosses some critical values, there appear certain resonances in the equation producing solutions that bifurcate from infinity. We study the bifurcation branches, and characterize when they are sub- or supercritical. Furthermore, we apply these results and techniques to obtain Landesman-Lazer type conditions guarantying the existence of solutions in the resonant case and to obtain a uniform Anti-Maximum Principle and several results related to the spectral behavior when the potential at the boundary is perturbed. We also characterize the stability type of the solutions in the unbounded branches.

In the remainder of this paper, we start our analysis on a sublinear oscillatory nonlinearity. We first focus our attention on the loss of Landesman-Lazer type conditions, and even in that situation, we are able to prove the existence of infinitely many resonant solutions and infinitely many turning points.

Next we focus our attention on stability switches. Even in the absence of resonant solutions, we are able to provide sufficient conditions for the existence of sequences of stable solutions, unstable solutions, and turning points.

We also discuss on bifurcation from the trivial solution set, and on a sublinear oscillatory nonlinearity.

Finally, we states a formula for the derivative of a localized Steklov eigenvalue on a subset of the boundary, with respect to tangential variations of that subset.

Keywords: Bifurcation from infinity, stability, instability, multiplicity, resonance, turning points.
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Bifurcación para un problema elíptico con condiciones de frontera no lineales

Resumen. Este artículo presenta un estudio sobre bifurcación para problemas elípticos con condiciones de frontera no-lineales. Consideramos una ecuación elíptica con condiciones de frontera no-lineales dependiendo de un parámetro. Supondremos que el término no lineal es asintóticamente lineal en el infinito. Cuando el parámetro cruza ciertos valores críticos (conocidos como los autovalores de Steklov) aparece un fenómeno de resonancia en la ecuación, lo que garantiza la existencia de ramas no acotadas de soluciones. Este fenómeno se conoce como bifurcación desde infinito. Estudiamos las ramas de soluciones y caracterizamos cuando son subcríticas (a la izquierda del autovalor) o supercríticas (a la derecha del autovalor). Aplicamos estos resultados para obtener condiciones del tipo Landesman-Lazer, que garantizan la existencia de soluciones para el problema resonante (cuando el parámetro coincide con el autovalor). Obtenemos también un Principio del Anti-Máximo, y resultados relativos al comportamiento espectral, cuando se perturba el potencial en la frontera. Además caracterizamos el tipo de estabilidad de las soluciones en dichas ramas no acotadas.

En el resto del articulo, analizamos no linealidades oscilatorias y sublineales. Centramos nuestra atención en la pérdida de condiciones del tipo Landesman-Lazer. Incluso en esta situación, demostramos la existencia de una sucesión de infinitas soluciones del problema resonante y una sucesión de infinitos puntos de retroceso.

A continuación, analizamos los cambios de estabilidad. Incluso en ausencia de soluciones resonantes, proporcionamos condiciones suficientes para la existencia de una sucesión de infinitas soluciones estables, una sucesión de infinitas soluciones inestables y una sucesión de infinitos puntos de retroceso.

También analizamos la bifurcación desde la solución trivial con una no-linealidad de tipo sublineal y oscilatorio.

Finalmente establecemos una fórmula para la derivada del autovalor de Steklov localizado sobre un subconjunto de la frontera, con respecto a variaciones tangenciales del subconjunto.

Palabras claves: Bifurcación en el infinito, estabilidad, inestabilidad, multiplicidad, resonancia, puntos de inflexión.

1. Introduction

In the last two decades a lot of attention has been payed to problems with nonlinear boundary conditions (see for instance [5] and references therein for parabolic problems with nonlinear boundary conditions with critically growing non-linearities). It is a natural question to analyze the dynamics and bifurcations induced by the nonlinear boundary conditions, and compare its effects with the case of an interior reaction term, which has been more widely studied. In this direction (see for example [6]) it is considered the existence of *patterns* for such problems, i.e., stable nontrivial equilibrium, see also the references therein for some previous and related results.

In this paper we consider the evolutionary equation of parabolic type with nonlinear boundary conditions depending of a parameter $\lambda \in \mathbb{R}$

$$\left\{\begin{array}{rcll} u_t - \Delta u + u &=& 0, & \text{in } \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} &=& \lambda u + g(x,u), & \text{on } \partial\Omega,\ t > 0, \\ u(0,x) &=& u_0(x), & \text{in } \Omega \end{array}\right. \tag{1}$$

in a bounded and sufficiently smooth domain $\Omega \subset \mathbb{R}^N$ with $N \geq 2$. We analyze the behavior and stability properties of the equilibrium solutions. These equilibria are solutions of the following elliptic problem

$$\left\{\begin{array}{rcll} -\Delta u + u &=& 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} &=& \lambda u + g(x,u), & \text{on } \partial\Omega. \end{array}\right. \tag{2}$$

This paper presents in a unified manner some of the recent work in this field. We concentrate our attention mostly, but not only, in [7, 8, 9, 11, 12, 29]. All the results presented here are essentially available for $g = g(\lambda, x, u)$, and in fact, in the references mentioned are written for such a nonlinearity g. We decide to present this survey for $g = g(x,u)$ by the shake of briefness. We send to those references for the interested reader.

The main hypothesis on the nonlinearity g is the sublinearity at infinity with respect to the variable u. We assume, roughly speaking, that $|g(x,u)| = o(|u|)$ as $|u| \to \infty$. Hence, the boundary condition is asymptotically linear at infinity, since the dominant term for large values of $|u|$ is the linear term λu. This condition means that in the boundary condition, the dominant term for $|u|$ large is the linear term λu. In this respect we call this boundary condition asymptotically linear. This includes the case where $g(x,u) = g(x)$.

It is well known that problem (2) has a (unique) solution if λ is not an eigenvalue of the problem

$$\left\{\begin{array}{rcll} -\Delta \Phi + \Phi &=& 0, & \text{in } \Omega, \\ \dfrac{\partial \Phi}{\partial n} &=& \sigma \Phi, & \text{on } \partial\Omega. \end{array}\right. \tag{3}$$

This eigenvalue problem is known as the *Steklov eigenvalue problem* and it is well known that (3) has a discrete set of eigenvalues $\{\sigma_i\}_{i=1}^{\infty}$. These numbers play an essential role in the analysis below. In particular, for $\lambda \notin \{\sigma_i\}_{i=1}^{\infty}$, we consider the operator T_λ such that $T_\lambda b := v$, where v is the unique solution of

$$\left\{\begin{array}{rcll} -\Delta v + v &=& 0, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} - \lambda v &=& b, & \text{on } \partial\Omega. \end{array}\right. \tag{4}$$

for a function b given on $\partial\Omega$.

The norm of the operator T_λ, for compact sets of λ far from the Steklov eigenvalues, is uniformly bounded, in some appropriate spaces. This fact joint with the sublinearity of the function g, allow us to show, by a fixed point argument, the existence of at least a solution of (2) for any λ not an Steklov eigenvalue. Moreover, all solutions are uniformly bounded for λ in compact intervals far from the Steklov eigenvalues (see Theorem 2.7).

On the other hand, when the parameter λ approaches an Steklov eigenvalue, the norm of the operator T_λ diverges to ∞. This fact is a first hint of the possibility of finding unbounded branches of solutions and reveals the resonant mechanism at the boundary that produces such large solutions. For instance, when $g \equiv 0$ the structure of the solutions of the problem (2) is well known: if λ is not an Steklov eigenvalue, the only solution is the trivial solution and if λ is an Steklov eigenvalue, the whole space of eigenfunctions associated to that eigenvalue are solutions of the elliptic problem which can be regarded as unbounded branches of solutions. For the case where g is sublinear at infinity, we will apply general techniques of bifurcation theory (see [15], [30], [31]) and will prove the existence of unbounded branches of solutions whenever the parameter λ approaches an Steklov eigenvalue of odd multiplicity (see Theorem 2.10). Moreover, since the first Steklov eigenvalue is simple, we will show the existence of unbounded branches of solutions bifurcating from the first eigenvalue. The fact that the first Steklov eigenfunction does not change sign will give us extra information that will permit us to analyze this branch of solutions in detail. In particular, we will show the existence of two branches of solutions one consisting of positive solutions and the other negative solutions (see Theorem 2.11).

This problem has already been studied in [7, 8] where we analyzed the existence of unbounded sets of solutions as well as their stability and some of the dynamical properties of the associated parabolic problem. This analysis was carried over assuming that the nonlinear term is sublinear at infinity. This assumption, by a mechanism of parametric resonance at the boundary, produces unbounded *branches* of solutions when λ approaches one of the Steklov eigenvalues of odd multiplicity. These branches *bifurcate from infinity* in the sense of [31, 30].

The set of solutions bifurcating at σ_1, the first Steklov eigenvalue, is made up of large positive solutions or large negative solutions (or both). We will denote by $\mathcal{D}^+ \subset \mathbb{R}\times C(\bar\Omega)$ (resp. $\mathcal{D}^- \subset \mathbb{R}\times C(\bar\Omega)$) the branch of positive, (resp. negative) solutions bifurcating at σ_1. As a matter of fact, the solutions in $\mathcal{D}^\pm$, can be described as

$$u = s\Phi_1 + w, \qquad \text{where} \quad w = o(|s|) \qquad \text{as } |s| \to \infty; \tag{5}$$

see Theorem 2.10.

Hereafter we will concentrate on the positive unbounded branch, $\mathcal{D}^+$ bifurcating at σ_1. The case for $\mathcal{D}^-$ is completely analogous.

In fact, for some continuum of solutions of (2), that we denote by u_λ, we have that

$$\frac{u_\lambda(x)}{\|u_\lambda\|_{L^\infty(\partial\Omega)}} \to \pm\Phi_1(x), \quad \text{in} \quad C^\beta(\overline{\Omega}\,) \quad \text{as} \quad \lambda \to \sigma_1, \tag{6}$$

for some $0 < \beta < 1$ and where $\Phi_1(x) > 0$ denotes the first positive Steklov eigenfunction, normalized in $L^\infty(\partial\Omega)$; see Corollary 3.2 in [7]. The choice of the sign depends on whether the *subbranch* is made of positive or negative equilibria. Note also that Φ_1 is strictly positive in $\overline{\Omega}$. In particular, from this we have

$$\inf_{x\in\overline{\Omega}} |u_\lambda(x)| \to \infty, \quad \text{as} \quad \lambda \to \sigma_1. \tag{7}$$

On the other hand, for λ far away from the Steklov eigenvalues, the set of solutions of (2) is nonempty and bounded in $\overline{\Omega}$, uniformly in λ. Also, as $\lambda \to \sigma_1$ equilibrium solutions that do not satisfy (6), remain bounded in $\overline{\Omega}$.

In the terminology of Bifurcation Theory, we say that, as $\lambda \to \sigma_1$, the unbounded branches of solutions of (2), u_λ, *bifurcate from infinity*, and that there exists a bifurcation from infinity at σ_1; cf. [31].

We proceed further in analyzing the structure and properties of unbounded branches of solutions of the elliptic problem (2) and on the global dynamics of the parabolic problem (1), when λ crosses σ_1. One important question is whether the bifurcating branch $\mathcal{D}^+$ is *subcritical* or *supercritical*. That is, if it is formed only with solutions (u) with $\lambda < \sigma_1$ or $\lambda > \sigma_1$ respectively.

To analyze this question, a condition on sublinearity of g is not enough to distinguish between the type of bifurcation and to accomplish this we will need to specify the precise asymptotics of the function g at infinity. For instance, if we consider that the function g behaves like $a|u|^\alpha$ as $u \to +\infty$, we can easily see that the sign of a will determine whether the bifurcation of positive solutions emanating from the first eigenvalue is sub or supercritical. For this, if $0 < u_n \to \infty$ is a solution of (2) for $\lambda_n \to \sigma_1$, multiplying the equation by the first Steklov eigenfunction $\Phi_1 > 0$ and integrating by parts we obtain

$$(\sigma_1 - \lambda_n) \int_{\partial\Omega} u_n \Phi_1 \, d\varsigma = \int_{\partial\Omega} g(x, u_n) \Phi_1 \, d\varsigma.$$

But since $u_n > 0$ and $u_n \to \infty$, then

$$\int_{\partial\Omega} u_n \Phi_1 \, d\varsigma > 0, \qquad \int_{\partial\Omega} g(x, u_n) \Phi_1 \, d\varsigma \approx a \int_{\partial\Omega} |u_n|^\alpha \Phi_1 d\varsigma,$$

and the sign of $\sigma_1 - \lambda_n$ is the same as the sign of a. Hence, if $a > 0$ the bifurcation of positive solutions will be subcritical and if $a < 0$, it will be supercritical (see Theorem 2.14 for a more general statement).

The *Maximum Principle* states a sign preserving property for the solutions of linear elliptic problems

$$\begin{cases} -\Delta u &= \lambda u + f(x), & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{cases} \tag{8}$$

when the parameter is less than the first eigenvalue: positive data $f > 0$, gives positive solution $u > 0$.

The *Anti-Maximum Principle* states a sign reversing property when the parameter crosses the first eigenvalue but still remains close to it: positive data $f > 0$, gives negative solution $u < 0$. In [13] Clement and Peletier prove the well known Anti-Maximum Principle for an elliptic problem with Dirichlet boundary conditions. In [3] Arcoya and Gamez generalize this result for the same problem, relaxing the hypothesis. It will be enough that $\int_\Omega f\varphi_1 > 0$, where $\varphi_1 > 0$ is the first eigenfunction

$$\begin{cases} -\Delta \varphi_1 &= \lambda \varphi_1, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega. \end{cases} \tag{9}$$

In [4] Arcoya and Rossi analyze the Anti-Maximum Principle for quasilinear problems.

In Section 3 we state and prove a uniform Anti-Maximum Principles for the problem (4) for varying potentials.

We will also show that some stability or instability of such solutions can be derived. We give conditions, which involve a more detailed knowledge of the behavior of the nonlinear term as $|u| \to \infty$, which imply that the unbounded branch of positive equilibria is subcritical, unique and stable (see Theorem 4.5). In an almost exact complementary situation, we also show that the unbounded branch of positive equilibria is supercritical, unique and unstable (see Theorem 4.6).

Another interesting question is that of the resonant problem, that is when $\lambda = \sigma_1$. For this case, we obtained in Theorem 5.1 of [7] some Landesman–Lazer type conditions guaranteeing that the resonant problem has solution; cf. [26]. In the language of bifurcation, these type of conditions can be stated as: if all the unbounded branches are either subcritical or supercritical then the resonant problem has at least one solution.

Therefore, in this paper we also consider nonlinearities sublinear and oscillatory. We hope to translate this oscillatory character of the nonlinear term at infinity, into an oscillatory behavior of the bifurcating branches. Observe that in this situation, both the criteria for sub/super criticality and the Landesman–Lazer type conditions do not hold.

In such a situation our goal is threefold: first we give easy–to–check conditions on the nonlinear term, guaranteeing that in $\mathcal{D}^+$ there are large subcritical and supercritical solutions.

Second, the connectedness of $\mathcal{D}^+$, suggests that we would be able to find an unbounded sequence of *turning points*, which are defined as

Definition 1.1. A solution (λ^*, u^*) of (2) in the branch of solutions $\mathcal{D}^+ \subset \mathbb{R} \times C(\bar{\Omega})$ is called a **turning point** if there is a neighborhood W of (λ^*, u^*) in $\mathbb{R} \times C(\bar{\Omega})$ such that, either $W \cap \mathcal{D}^+ \subset [\lambda^*, \infty) \times C(\bar{\Omega})$ or $W \cap \mathcal{D}^+ \subset (-\infty, \lambda^*] \times C(\bar{\Omega})$.

Note that, generically, in a neighborhood of a turning point there are, at least, two solutions for the same value of the parameter *at one side*, either $\lambda < \lambda^*$, or either for $\lambda > \lambda^*$. Therefore, turning points are related with multiplicity of solutions.

Third, the connectedness of $\mathcal{D}^+$, suggests that we would be able to find an unbounded sequence of resonant solutions. Let us remark that this result on infinitely many resonant solutions is attained when the Landesman-Lazer conditions do not hold.

The paper is organized in the following way. In Section 2 is stated the framework and the bifurcation from infinity results. Specifically, it contains Theorem 2.7 on existence of bounded solutions, Theorem 2.10 on bifurcation from infinity, Theorem 2.11, Theorem 2.14, and Theorem 2.16 on bifurcation from infinity from a simple eigenvalue, from the first and from higher eigenvalues respectively.

In Section 3 is stated an Anti-Maximum Principle and also a uniform Anti-Maximum Principle (see Theorem 3.1 and Theorem 3.4 respectively).

Section 4 is devoted to the stability analysis of the solutions, see Theorem 4.5, Theorem 4.6, on the stability (unstability) of the solutions in the unbounded branch.

In Section 5 we discuss on the resonant case, see Theorem 5.1 on Landesman–Lazer type conditions providing the existence of at least a resonant solution. We start our analysis on a sublinear oscillatory nonlinearity, focusing firstly our attention on the existence of infinitely many resonant solutions and infinitely many turning points (see Theorem 5.6).

In Section 6 we continue with our study of sublinear oscillatory nonlinearity, now focusing our attention on stability switches, concretely Theorem 6.1 and Theorem 6.5 provides sufficient conditions for the existence of infinitely many turning points, possibly without resonant solutions.

In Section 7 we discuss on bifurcation from the trivial solution set (see Theorem 7.1). See also Theorem 7.3 and Theorem 7.4 on oscillatory nonlinearity sublinear at zero.

Finally, Section 8 states a formula for the derivative of a localized Steklov eigenvalue with respect to tangential variations. We end this paper with bibliographical notes.

2. Bifurcation from infinity

To start with, by solutions to (2) we mean elements $u \in H^1(\Omega)$ such that the weak formulation holds, i.e.

$$\int_\Omega (\nabla u \cdot \nabla v + uv)\, dx = \lambda \int_{\partial\Omega} uv\, d\sigma + \int_{\partial\Omega} g(x,u)v\, d\sigma \quad \text{for all} \quad v \in H^1(\Omega). \tag{10}$$

We will show that, as λ approaches some eigenvalue, there exists an unbounded branch of solutions of (2), u_λ. As stated in Theorem 2.10, due to (H2) there exists a connected set of positive solutions of (2). We denote it by $\mathcal{D}^+ \subset \mathbb{R} \times C(\bar{\Omega})$, and recall that for $(\lambda, u_\lambda) \in \mathcal{D}^+$

$$u = s\Phi_1 + w, \qquad \text{with} \quad w = o\,(|s|) \quad \text{and} \quad |\sigma_1 - \lambda| = o(1) \quad \text{as } |s| \to \infty.$$

The set $\mathcal{D}^+$ is known as a *branch bifurcating from infinity* in the sense of Rabinowitz (cf. [31, 7]).

In the terminology of Bifurcation Theory, there is a branch of solutions *bifurcating from infinity* (see [31]).

There are many works studying bifurcation from infinity for related problems (cf. [30, 31] for an abstract framework). A similar analysis for the case of an interior reaction term was first established in [3].

This section is organized as follows. In Subsection 2.1 we formulate the problem and show the existence of solutions for all values of the parameter λ different from the Steklov eigenvalues. To accomplish this, we analyse the associated linear problem, stating and proving several important regularity results. Then, we formulate the nonlinear problem (2) as a fixed point problem in certain function space on the boundary. Finally, the compactness results obtained through the regularity results and the Schaeffer fixed point theorem will show the existence of solutions.

In Subsection 2.2 we apply bifurcation results, mainly from [30, 31], to show the existence of unbounded branches of solutions bifurcating from the Steklov eigenvalues (see Theorem 2.10). We pay special attention to the bifurcations emanating from simple eigenvalues (see Theorem 2.11).

In Subsection 2.3 we give conditions on the nonlinearity g that allows us to characterize the type bifurcations, sub or supercritical.

Finally, we also consider the one dimensional case, that is, where the equation (2) is possed in $\Omega = (0,1) \subset \mathbb{R}$.

2.1. Setting of the problem

Throughout this paper we will assume:

(H1) $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathèodory function (i.e. $g = g(x,s)$ is measurable in $x \in \Omega$, and continuous with respect to $s \in \mathbb{R}$). Moreover, there exist $G_1 \in L^r(\partial\Omega)$ with $r > N-1$ and a continuous function $U : \mathbb{R} \to \mathbb{R}^+$, satisfying

$$\begin{cases} \|g(x,s)| \leq G_1(x)U(s), & \forall (x,s) \in \partial\Omega \times \mathbb{R}, \\ \lim\limits_{|s|\to\infty} \dfrac{U(s)}{s} = 0, & \end{cases}$$

which in turn it implies that

$$\limsup_{|s|\to\infty} \left| \frac{g(x,s)}{s} \right| \to 0,$$

that is, the function g is sublinear at infinity in the variable s.

In this Subsection we rewrite equation (2) as a fixed point problem in appropriate function spaces and analyze the existence of solutions for all $\lambda \in \mathbb{R}$ except for a discrete set, the eigenvalue set. To accomplish this task we will use Schaeffer's fixed point theorem (cf. [17]).

With respect to the linear problem, it is already well known (cf. [2]), that the operator $A = -\Delta + I$, with homogeneous Neumann boundary conditions defines an unbounded operator in $L^p(\Omega)$ for all $p > 1$ with domain

$$D(A) = \{u \in W^{2,p}(\Omega); \partial u/\partial n = 0 \qquad \text{in} \quad \partial\Omega\}.$$

Moreover, the operator A has an associated scale of interpolation-extrapolation spaces and, in particular, for each $p > 1$, we have that

$$A : W^{1,p}(\Omega) \to W^{-1,p}(\Omega) \qquad \text{is an isomorphism.}$$

Moreover, for any $q \geq 1$, the embedding

$$L^q(\partial\Omega) \hookrightarrow W^{-1,p}(\Omega) \qquad \begin{cases} \text{is continuous for} & p = \dfrac{qN}{N-1}, \\ \text{and compact if} & p < \dfrac{qN}{N-1}. \end{cases}$$

Hence, we have that for $b \in L^q(\partial\Omega)$, the unique solution of the elliptic problem with nonhomogeneous Neumann boundary condition

$$\left\{ \begin{array}{rcll} -\Delta v + v & = & 0, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} & = & b, & \text{on } \partial\Omega, \end{array} \right. \tag{11}$$

is given by

$$v = A^{-1}(b) \in W^{1,p}(\Omega),$$

and moreover

$$\|v\|_{W^{1,p}(\Omega)} \le C\|b\|_{L^q(\partial\Omega)}.$$

We will denote the operators

$$T(b) = v \qquad \text{and} \quad S(b) = \gamma T(b), \qquad \text{where} \quad \gamma \qquad \text{is the } \textit{trace} \text{ operator.}$$

The operator S is known as the *Neumann-to-Dirichlet operator*. Hence, the operator T takes functions defined on $\partial\Omega$ to functions defined in Ω and S takes functions defined on $\partial\Omega$ to functions defined on $\partial\Omega$.

Our first task will be to show that any weak solution $u \in H^1(\Omega)$ of (2) lies in the more regular space $C^\alpha(\bar{\Omega})$. To accomplish this, we will need several regularity results of the associated linear problems. As a matter of fact, as a consequence of the above, and using embedding and trace theorems we can easily show the following regularity results,

Lemma 2.1. *If $N \ge 2$ and $b \in L^q(\partial\Omega)$ with $q \ge 1$, then, the solution $v = Tb$ of (11) satisfies*

$$v \in W^{1,p}(\Omega) \qquad \textit{for} \quad 1 \le p \le \frac{qN}{N-1}, \qquad \textit{with} \quad \|v\|_{W^{1,p}(\Omega)} \le C\|b\|_{L^q(\partial\Omega)}.$$

In particular, we have

(i) If $1 \le q < N-1$, then

$$\gamma v \in L^r(\partial\Omega) \qquad \textit{for all} \quad 1 \le r \le \frac{(N-1)q}{N-1-q},$$

and the map

$$S : L^q(\partial\Omega) \to L^r(\partial\Omega) \qquad \begin{cases} \textit{is continuous for} & 1 \le r \le \dfrac{q(N-1)}{N-1-q}, \\ \textit{and compact if} & 1 \le r < \dfrac{q(N-1)}{N-1-q}. \end{cases}$$

(ii) If $q = N-1$, then

$$\gamma v \in L^r(\partial\Omega) \qquad \textit{for all} \quad r \ge 1,$$

and the map $S : L^q(\partial\Omega) \to L^r(\partial\Omega)$ is continuous and compact for $1 \le r < \infty$.

(iii) If $q > N-1$, then

$$v \in C^\alpha(\overline{\Omega}) \qquad \textit{with} \quad \|v\|_{C^\alpha(\overline{\Omega})} \le C\|b\|_{L^q(\partial\Omega)} \qquad \textit{for some} \quad \alpha \in (0,1);$$

moreover, $\gamma v \in C^\alpha(\partial\Omega)$ and the map $S : L^q(\partial\Omega) \to C^\alpha(\partial\Omega)$ is continuous and compact.

Proof. We only have to take into account the above, that the trace operator

$$\gamma : W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega),$$

and that the Sobolev imbedding Theorems for noninteger order, state that

$$\begin{cases} \text{If } sp < N-1, \ W^{s,p}(\partial\Omega) \hookrightarrow L^r(\partial\Omega), \text{ with continuous imbedding for } r \leq p^*, \\ \qquad \text{where } \ \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{s}{N-1}, \qquad \text{and compact imbedding for } \quad r < p^*. \\ \\ \text{If } sp = N-1, \ W^{s,p}(\partial\Omega) \hookrightarrow L^r(\partial\Omega), \text{ with } 1 \leq r < \infty. \\ \text{If } sp > N-1, \ W^{s,p}(\partial\Omega) \hookrightarrow C^{m,\alpha}(\overline{\partial\Omega}), \text{ with continuous imbedding for} \\ \qquad \alpha = s - \dfrac{N-1}{p} - m \ \text{ and compactly imbedded in } \quad C^{m,\beta}(\overline{\partial\Omega}) \qquad \text{for} \quad \beta < \alpha, \end{cases}$$

cf. [1]. ☑

As an immediate corollary, we have the following technical result,

Corollary 2.2. *i) If $b \in L^q(\partial\Omega)$ for any $q \geq 1$, then*

$$Sb \in L^{q+\frac{1}{N}}(\partial\Omega).$$

(ii) Assume that b satisfies

$$|b(x)| \leq h(x)w(x) \qquad \textit{where} \quad h \in L^r(\partial\Omega) \qquad \textit{with} \quad r > N-1.$$

Let us define $\delta = \frac{N-1}{N-2} - r' > 0$. If $w \in L^p(\partial\Omega)$ with $\frac{1}{N-1} \leq \frac{1}{p} + \frac{1}{r} \leq 1$, then

$$Sb := \gamma v \in L^{p+\delta}(\partial\Omega) \qquad \textit{and} \quad \|Sb\|_{L^{p+\delta}(\partial\Omega)} \leq C\|w\|_{L^p(\partial\Omega)}.$$

Proof. (i) Observe that if $q \geq N-1$, then from the above Corollary $\gamma v \in L^r(\partial\Omega)$ for all $r \geq 1$. In case $1 \leq q < N-1$, then $Sb \in L^r(\partial\Omega)$ for $r \leq \frac{(N-1)q}{N-1-q}$. A simple computation shows that

$$\frac{(N-1)q}{N-1-q} - q \geq \frac{1}{N}, \text{ for } 1 \leq q < N-1.$$

(ii) Notice that $hw \in L^{pr/(p+r)}(\partial\Omega)$ and $\frac{pr}{p+r} \geq 1$ because $\frac{1}{p} + \frac{1}{r} \leq 1$. Hence, by Lemma 2.1 $\gamma v \in L^s(\partial\Omega)$ with $s = \frac{\frac{pr}{p+r}(N-1)}{N-1-pr/(p+r)}$. If we denote by $y = \frac{pr}{p+r} = \frac{1}{\frac{1}{p}+\frac{1}{r}}$, then $1 \leq y \leq N-1$, $p = \frac{ry}{r-y}$ and

$$\min_{\frac{1}{N-1} \leq \frac{1}{p}+\frac{1}{r} \leq 1} \left\{ \frac{\frac{pr}{p+r}(N-1)}{N-1-pr/(p+r)} - p \right\} = \min_{1 \leq y \leq N-1} \left\{ \frac{y(N-1)}{N-1-y} - \frac{ry}{r-y} \right\}.$$

But a simple computation shows that this last minimum is attained at $y = 1$. This concludes the proof of the Corollary. ☑

These regularity results with a bootstrap argument will allow us to prove the following

Proposition 2.3. *Assume g satisfies* (H1). *Let us fix any $R > 0$. If $u \in H^1(\Omega)$ is a solution of* (2) *for some $|\lambda| \leq R$, then, we have*

$$\|u\|_{C^\alpha(\bar{\Omega})} \leq C(1 + \|u\|_{L^p(\partial\Omega)}) \tag{12}$$

for some positive α, where $C = C(R)$ and $p = 2(N-1)/(N-2)$.

Proof. Assume $N \geq 3$ (the proof when $N = 2$ is simpler). Observe that the boundary condition satisfied by u is $\frac{\partial u}{\partial n} = \lambda u + g(x, u)$, and by hypothesis (H1) we have

$$|g(x, u)| \leq CG_1 x)(1 + |u(x)|) \qquad \text{for some constant} \quad C = C(R).$$

Hence $\frac{\partial u}{\partial n} = b(x)$ with $|b(x)| \leq C(1 + G_1 x))(1 + |u(x)|)$. Notice also that $1 + G_1 \in L^r(\partial\Omega)$ for some $r > N - 1$.

Now, if $u \in H^1(\Omega)$, then $\gamma u \in L^p(\partial\Omega)$ with $p = 2\frac{N-1}{N-2}$ which satisfies that $\frac{1}{p} + \frac{1}{r} \leq 1$ for any $r > N-1$. Hence, $b \in L^s(\partial\Omega)$ with $\frac{1}{r} + \frac{1}{p+\delta} = \frac{1}{s}$; if $s > N - 1$, then Lemma 2.1 (iii) implies that $u \in C^\alpha(\bar{\Omega})$ and

$$\|u\|_{C^\alpha(\bar{\Omega})} \leq C\|b\|_{L^s(\partial\Omega)} \leq C(1 + \|u\|_{L^p(\partial\Omega)}).$$

If $s \leq N - 1$, applying the regularity result of Corollary 2.2 (ii) we obtain that $\gamma u \in L^{p+\delta}(\partial\Omega)$ and

$$\|u\|_{L^{p+\delta}(\partial\Omega)} \leq C(1 + \|u\|_{L^p(\partial\Omega)}) \tag{13}$$

Certainly, for a finite k we will have

$$\frac{1}{p + (k-1)\delta} + \frac{1}{r} \geq \frac{1}{N-1} \qquad \text{and} \quad \frac{1}{p + k\delta} + \frac{1}{r} < \frac{1}{N-1}.$$

Repeating this regularity argument k times, we get that $\gamma u \in L^{p+k\delta}(\partial\Omega)$. Moreover, we will also have

$$\|u\|_{L^{p+k\delta}(\partial\Omega)} \leq C(1 + \|u\|_{L^{p+(k-1)\delta}(\partial\Omega)}) \leq \cdots \leq C(1 + \|u\|_{L^p(\partial\Omega)}).$$

In particular, $b \in L^s(\partial\Omega)$ for some $s > N - 1$. Hence, Lemma 2.1 (iii) and (13) finish the proof. ☑

Remark 2.4. The regularity result of the Proposition 2.3 tells us that looking for solutions of problem (2) in $H^1(\Omega)$ is equivalent to looking for solutions in a more regular space like $C^\alpha(\bar{\Omega})$.

We analyze now the operator S, the Neumann-to-Dirichlet operator. We have the following result,

Lemma 2.5. *The operator $S : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is a linear selfadjoint, positive and compact operator. If we denote by $\{\tau_i\}_{i=1}^\infty$ its eigenvalues, and by $\sigma_i = 1/\tau_i$ we have that for any $\lambda \in \mathbb{R}$, $\lambda \notin \{\sigma_i\}_{i=1}^\infty$, the operator $S_\lambda : L^2(\partial\Omega) \to L^2(\partial\Omega)$ defined by $S_\lambda(g) = \gamma v$ where v is the unique solution of*

$$\begin{cases} -\Delta v + v = 0, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} - \lambda v = g, & \text{on } \partial\Omega, \end{cases} \tag{14}$$

is selfadjoint, continuous and compact. Moreover, the first eigenvalue σ_1 is simple and its eigenfuction Φ_1 can be choosen strictly positive. Also, if $r > N-1$ then, $S_\lambda : L^r(\partial\Omega) \to C^0(\partial\Omega)$ is continuous and compact and for any compact set $K \subset \mathbb{R} \setminus \{\sigma_i\}_{i=1}^\infty$ the norm of $S_\lambda : L^r(\partial\Omega) \to C^0(\partial\Omega)$ is uniformly bounded for $\lambda \in K$. Also, $\|S_\lambda\| \to \infty$ as $\lambda \to \sigma_i$ for some i.

Proof. Observe that if b_1, $b_2 \in L^2(\partial\Omega)$, and if v_1, v_2 are the solutions of $-\Delta v_i + v_i = 0$ in Ω, $\frac{\partial v_i}{\partial n} = b_i$, $i = 1, 2$, then by the weak formulation of this problem we have that

$$(S(b_1), b_2)_{L^2(\partial\Omega)} = \int_\Omega \nabla v_1 \nabla v_2 + \int_\Omega v_1 v_2 = (b_1, S(b_2))_{L^2(\partial\Omega)}. \tag{15}$$

From (15) it follows that S is selfadjoint and positive. That S is compact follows from Lemma 2.1. The fact that the first eigenfunction can be choosen nonnegative follows easily from the Rayleigh quotient for the first eigenvalue. Then, maximum principles imply that the first eigenfunction is actually strictly positive. In turn, this implies that the first eigenvalue is simple.

The rest of the proof follows just by realizing that $S_\lambda = (I - \lambda S)^{-1} \circ S$ and applying the regularity results of Corollary 2.2. ☑

It is clear now that we can set a fixed point problem to obtain the solutions of (2). As a matter of fact, $u \in H^1(\Omega)$ is a solution of (2) if and only if its trace $v = \gamma u$ is a fixed point of

$$v = \lambda S v + S(g(\cdot, v)). \tag{16}$$

Notice also that once v is obtained we recover u by solving $-\Delta u + u = 0$ in Ω with $u = v$ on the boundary.

Concerning the fixed point problem (16), we have

Lemma 2.6. *Under hypotheses* (H1)*, the map $C^0(\partial\Omega) \ni v \to g(\cdot, v) \in L^r(\partial\Omega)$ is well defined and continuous. Moreover, for each $\varepsilon > 0$, there exists a constant $C = C(\varepsilon,)$ such that*

$$\|g(\cdot, v)\|_{L^r(\partial\Omega)} \le \varepsilon \|v\|_{C^0(\partial\Omega)} + C \tag{17}$$

for all $v \in C^0(\partial\Omega)$.

In particular, the map $C^0(\partial\Omega) \ni v \to S_\lambda(g(\cdot, v)) \in C^0(\partial\Omega)$, is continuous and compact for all $\lambda \in \mathbb{R} \setminus \{\sigma_i\}_{i=1}^\infty$.

Proof. That this map is well defined follows from the bounds of g given by (H1). The continuity follows from the continuity of g with respect to the last variable, the bounds of g given by (H1) and the dominated convergence theorem. Statement (17) follows from the fact that for each $\varepsilon > 0$ we have the inequality $|U(s)| \le \varepsilon s + C$, for some constant $C = C(\varepsilon)$.

The last part of the lemma follows easily. ☑

Now we are in a position where we can show the existence of solutions of our original problem (2) for all $\lambda \in \mathbb{R} \setminus \{\sigma_i\}_{i=1}^\infty$. We have the following

Theorem 2.7. *If g satisfies* (H1) *then, for all $\lambda \in \mathbb{R} \setminus \{\sigma_i\}_{i=1}^{\infty}$ there exists at least one solution of problem* (2). *Moreover, for each compact set $K \subset \mathbb{R} \setminus \{\sigma_i\}_{i=1}^{\infty}$, we have the existence of a constant $C = C(K)$ such that any solution of problem* (2) *is bounded in $C^0(\Omega)$ by C.*

Proof. Consider the compact set $K \subset \mathbb{R}\setminus\{\sigma_i\}_{i=1}^{\infty}$ and observe that by Lemma 2.5 we have that there exists a constant $C_1 = C_1(K)$ such that the norm of $S_\lambda : L^r(\partial\Omega) \to C^0(\Omega)$ is bounded by C_1 for all $\lambda \in K$.

We will apply Schaeffer fixed point argument to (16) (cf. [17]). For this, consider $\theta \in [0, 1]$ and let v be a fixed point of

$$v = \theta S_\lambda(g(\cdot, v)) \tag{18}$$

for some $\lambda \in K$. Then, $\|v\|_{C^0(\partial\Omega)} \leq C_1\|g(\cdot, v\|_{L^r(\partial\Omega)}$. But, by (17) we get

$$\|v\|_{C^0(\partial\Omega)} \leq C_1(\varepsilon\|v\|_{C^0(\partial\Omega)} + C(\varepsilon, K)).$$

Choosing ε small enough such that $1 - C_1\varepsilon \geq 1/2$, we get $\|v\|_{C^0(\partial\Omega)} \leq 2C_1C(\varepsilon, K)$. Noticing that by Lemma 2.6 we have that $v \to S_\lambda(g(\cdot, v)$ is compact in $C^0(\partial\Omega)$ when $\lambda \notin \{\sigma_i\}_{i=1}^{\infty}$, and applying Schaeffer fixed point argument, we prove the proposition. ☑

2.2. Unbounded branches of equilibria

From the previous results, it is clear that when the value of the parameter λ is away from the Steklov eigenvalues, the solutions of (2) are bounded uniformly in λ. On the other hand, since the norm of the operator S_λ blows up to infinity when λ approaches a Steklov eigenvalue (see Lemma 2.5), it is natural to expect the existence of branches of solutions that diverge to infinity in certain norms when the parameter approaches a Steklov eigenvalue. For instance, if we consider the case where $g \equiv 0$, then, for any $\lambda \notin \{\sigma_i\}_{i=1}^{\infty}$ the unique solution is $u \equiv 0$; while for $\lambda = \sigma_i$ we have that any function of the whole finite dimensional subspace given by the eigenfunctions associated to σ_i is a solution. This subspace constitutes an unbounded branch of solutions.

Let us start by analyzing the behavior of the solutions when we know explicitly that the solution blows up in certain norm.

Proposition 2.8. *Assume $\{\lambda_n\}_{n=1}^{\infty}$ is a convergent sequence of real numbers for which there exist solutions u_n of* (2) *with $\|u_n\|_{L^\infty(\partial\Omega)} \to \infty$ as $n \to \infty$. Then, necessarily $\lambda_n \to \sigma_i$ for certain $i \in \mathbb{N}$, and for any subsequence of u_n there exists another subsequence, that we denote by $u_{n'}$, and an eigenfunction Φ_i associated to σ_i with $\|\Phi_i\|_{L^\infty(\partial\Omega)} = 1$, such that*

$$\frac{u_{n'}}{\|u_{n'}\|_{L^\infty(\partial\Omega)}} \to \Phi_i, \quad \text{in} \quad C^\beta(\bar{\Omega}),$$

for some $\beta > 0$.

Proof. Applying the Hölder estimate given by (12) we obtain that if $v_n = u_n/\|u_n\|_{L^\infty(\partial\Omega)}$, we have $\|v_n\|_{C^\alpha(\bar{\Omega})} \leq C$, for some C independent of n. Using the compact embedding $C^\alpha(\bar{\Omega}) \hookrightarrow C^\beta(\bar{\Omega})$ for $0 < \beta < \alpha$, we obtain that for any subsequence of v_n, there exists another subsubsequence, $v_{n'}$ and a function $\Phi \in C^\beta(\bar{\Omega})$ such that $v_{n'} \to \Phi$ in $C^\beta(\bar{\Omega})$.

Therefore, since $\|v_{n'}\|_{L^\infty(\partial\Omega)} = 1$ we get that $\|\Phi\|_{L^\infty(\partial\Omega)} = 1$, and in particular Φ is not identically zero.

The equation satisfied by $v_{n'}$ is

$$\begin{cases} -\Delta v_{n'} + v_{n'} &= 0, & \text{in } \Omega,\\ \dfrac{\partial v_{n'}}{\partial n} &= \lambda_{n'} v_{n'} + \frac{g(x,u_{n'})}{\|u_{n'}\|_{L^\infty(\partial\Omega)}}, & \text{on } \partial\Omega. \end{cases}$$

Passing to the limit in the weak formulation of this equation, taking into account that $\frac{g(x,u_{n'})}{\|u_{n'}\|_{L^\infty(\partial\Omega)}} \to 0$ in $L^r(\partial\Omega)$ as $n' \to \infty$ and that $v_{n'} \to \Phi$, we get that Φ is a solution of

$$\begin{cases} -\Delta \Phi + \Phi &= 0, & \text{in } \Omega,\\ \dfrac{\partial \Phi}{\partial n} &= \sigma\Phi, & \text{on } \partial\Omega, \end{cases} \tag{19}$$

where $\sigma = \lim_{n'\to\infty} \lambda_{n'}$. This eigenvalue problem is known as the *Steklov eigenvalue problem.* Since $\|\Phi\|_{L^\infty(\partial\Omega)} = 1$, necessarily σ is an Steklov eigenvalue and Φ is an Steklov eigenfunction associated to σ. This proves the Proposition. ☑

We immediately have

Corollary 2.9. *With the same hypotheses of Proposition 2.8 we have*

(i) The whole sequence satisfies $\|u_n\|_{L^p(\partial\Omega)} \to \infty$ *for any* $1 \le p \le \infty$.

(ii) If $u_n \ge 0$ *for all* n, *then necessarily* $\lambda_n \to \sigma_1$ *and the whole sequence satisfies*

$$\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}} \to \Phi_1, \quad in \quad C^\beta(\bar\Omega).$$

Proof. (i) Since $L^p(\partial\Omega) \hookrightarrow L^1(\partial\Omega)$, it will be enough to show the result for $p = 1$. If this is not the case, then there will exist a subsequence u_n bounded in $L^1(\partial\Omega)$. From Proposition 2.8, we can get another subsequence $u_{n'}$ satisfying $u_{n'}/\|u_{n'}\|_{L^\infty(\partial\Omega)} \to \Phi_i$ and in particular $\|u_{n'}\|_{L^1(\partial\Omega)}/\|u_{n'}\|_{L^\infty(\partial\Omega)} \to \|\Phi_i\|_{L^1(\partial\Omega)} > 0$, which implies that $\|u_{n'}\|_{L^1(\partial\Omega)} \to \infty$, which is a contradiction.

(ii) From Proposition 2.8, any possible convergent subsequence of $u_n/\|u_n\|_{L^\infty(\partial\Omega)}$ has to converge to an Steklov eigenfunction Φ_i with $\|\Phi_i\|_{L^\infty(\partial\Omega)} = 1$. Since in this case $u_n \ge 0$, we have that $\Phi_i \ge 0$. But σ_1 is the unique Steklov eigenvalue with a nonnegative eigenfunction Φ_1 (see Lemma 2.5). ☑

We will show now that any Steklov eigenvalue σ of odd multiplicity is a bifurcation point from infinity, that is, there exists a sequence λ_n with $\lambda_n \to \sigma$ and a sequence of solutions u_n of (2) for the value λ_n such that $\|u_n\|_{L^\infty(\Omega)} \to \infty$.

Before stating the result, consider the following notation. We will consider the solutions of (2) in $\mathbb{R} \times C(\bar\Omega)$, where the first coordinate is the value of λ and the second is the function u, which is a solution of (2) for this value of λ. In this sense, we will denote the set of solutions by $\mathcal{S}$. Recall also that we have denoted the Steklov eigenvalues (eigenvalues of problem (3)) by $\{\sigma_i\}_{i=1}^\infty$.

We have the following result (cf. [31], Theorem 1.6):

Theorem 2.10. *Consider problem* (2) *and assume that the nonlinearity g satisfies conditions* (H1). *If σ is an Steklov eigenvalue of odd multiplicity, then the set of solutions of* (2)*, denoted by $\mathcal{S}$, possesses an unbounded component $\mathcal{D}$ which meets $(\sigma,\infty)\in\mathbb{R}\times C(\Omega)$.*

Moreover, if $[\lambda_-,\lambda_+]\subset\mathbb{R}$ is an interval such that $[\lambda_-,\lambda_+]\cap\{\sigma_i\}_{i=1}^{\infty}=\{\sigma\}$ and $\mathcal{M}=[\lambda_-,\lambda_+]\times\{u\in C(\bar\Omega):\|u\|_{C(\bar\Omega)}\geq 1\}$, then either

(i) *$\mathcal{D}\backslash\mathcal{M}$ is bounded in $\mathbb{R}\times C(\bar\Omega)$, in which case $\mathcal{D}\backslash\mathcal{M}$ meets the set $\{(\lambda,0),:\ \lambda\in\mathbb{R}\}$ at $(\lambda_0,0)$ such that $g(\lambda_0,\cdot,0)=0$, or*

(ii) *$\mathcal{D}\setminus\mathcal{M}$ is unbounded in $\mathbb{R}\times C(\bar\Omega)$.*

If $\mathcal{D}\backslash\mathcal{M}$ is unbounded, and it has a bounded projection on $\mathbb{R}$, then $\mathcal{D}\backslash\mathcal{M}$ meets $(\tilde\sigma,\infty)\in\mathbb{R}\times C(\bar\Omega)$, with $\sigma\neq\tilde\sigma\in\{\sigma_i\}_{i=1}^{\infty}$, i.e. $\mathcal{D}\backslash\mathcal{M}$ meets another bifurcation point from infinity.

Proof. We apply the general techniques from [31] to the fixed point problem (16) in the space $C(\partial\Omega)$. Thus, we have to prove that

(A) $S(g(\cdot,v))=o(\|v\|)$ at $v=\infty$ uniformly for λ in bounded intervals,and

(B) the map $(\lambda,v)\to\|v\|^2S(g(\cdot,v/\|v\|^2))$ is compact for λ in bounded intervals,

where for simplicity we denote by $\|v\|:=\|v\|_{C(\partial\Omega)}$.

(A) For any $v\in C(\partial\Omega)$ we have, from (H1), that $g(\cdot,v)\in L^r(\partial\Omega)$. Therefore,

$$\frac{\|S(g(\cdot,v))\|}{\|v\|}\leq C\frac{\|g(\cdot,v)\|_{L^r(\partial\Omega)}}{\|v\|}\leq C\left(\varepsilon+\frac{C_\varepsilon}{\|v\|}\right), \tag{20}$$

where we have used Lemma 2.1 for the first inequality and Lemma 2.6 for the second one. From (20) we easily get (A).

(B) We have to verify that $H:\mathbb{R}\times C(\partial\Omega)\to C(\partial\Omega)$ defined by

$$H(\lambda,v):=\|v\|^2S(g(x,v/\|v\|^2))\qquad\text{is compact.}$$

Note first that the image of

$$\{(\lambda,v)\in[\underline{\lambda},\overline{\lambda}]\times C(\partial\Omega):\ \delta\leq\|v\|_{C(\partial\Omega)}\leq\rho\}$$

under H is relatively compact for any $\underline{\lambda}<\overline{\lambda}$ and $0<\delta\leq\rho<\infty$. This follows from the boundedness of g and the compactness of S. Thus we only need to prove that the image of $[\underline{\lambda},\overline{\lambda}]\times B_\delta$ under H is relatively compact in $C(\partial\Omega)$ for some $\delta>0$ small enough, where $B_\delta:=\{v\in C(\partial\Omega):\ \|v\|\leq\delta\}$. Let us choose $v\in B_\delta$, and define $w=\dfrac{v}{\|v\|^2}$, which satisfies $\|w\|\geq\dfrac{1}{\delta}$.

From (17) with $\varepsilon=1$, we get

$$\frac{\|g(\cdot,w)\|_{L^r(\partial\Omega)}}{\|w\|}\leq C, \tag{21}$$

with $C = C(\lambda, \|h\|_{L^r(\partial\Omega)}, \delta)$. Therefore

$$\|v\|^2 \left\| g\left(\cdot, \frac{v}{\|v\|^2}\right)\right\|_{L^r(\partial\Omega)} \le C\|v\| \le C\delta. \tag{22}$$

Now, the compactness of $S : L^r(\partial\Omega) \to C(\partial\Omega)$ given by Lemma 2.1 ends the proof. ☑

We analyze now the case where the eigenvalue σ is simple, and in particular the case of the first eigenvalue. We have the following,

Theorem 2.11. *Let σ denote a simple Steklov eigenvalue and Φ a corresponding eigenfunction. Assume g satisfies hypothesis* (H1). *Then, the set of solutions of* (2)*, possesses two unbounded components $\mathcal{D}^+$, $\mathcal{D}^-$ which meet $(\sigma, \infty) \in \mathbb{R} \times C(\bar\Omega)$, satisfying*

(i) *there exists a neighborhood $\mathcal{O}_1$ of (σ, ∞) such that $(\lambda, u) \in \mathcal{D}^+ \cap \mathcal{O}_1$ and $(\lambda, u) \neq (\sigma, \infty)$ implies*

$$u = s\Phi + w \qquad \text{where } \; s > 0, \qquad \text{with } \|w\|_{L^\infty(\partial\Omega)} = o(|s|) \text{ at } |s| = \infty; \tag{23}$$

(ii) *there exists a neighborhood $\mathcal{O}_2$ of (σ, ∞) such that $(\lambda, u) \in \mathcal{D}^- \cap \mathcal{O}_2$ and $(\lambda, u) \neq (\sigma, \infty)$ implies*

$$u = -s\Phi + w \qquad \text{where } \; s > 0, \qquad \text{with } \|w\|_{L^\infty(\partial\Omega)} = o(|s|) \text{ at } |s| = \infty. \tag{24}$$

Proof. See [31], Corollary 1.8. ☑

Note, in particular, that if $\sigma = \sigma_1$, since the first eigenfunction can be chosen positive, this result implies the existence of branches of positive and negative solutions bifurcating from infinity.

2.3. Subcritical and supercritical bifurcations from infinity

In this subsection we give conditions on the nonlinearity g that allows us to characterize the type bifurcations, sub or supercritical (on the *left* or on the *right* of the first eigenvalue respectively) occurring.

As an example, let us consider the case where $v_n \to \Phi_1$ and assume, for instance, that the function $g(x, s)$ behaves for $s \to +\infty$ as

$$g(x, s) \approx G(x)s^\alpha.$$

Then, considering equation (2) with $\lambda = \lambda_n$, multiplying it by Φ_1, integrating by parts and using that Φ_1 is an eigenfunction, we get

$$(\sigma_1 - \lambda_n)\int_{\partial\Omega} u_n\Phi_1 = \int_{\partial\Omega} g(x, u_n)\Phi_1.$$

Hence, since $u_n \to +\infty$ uniformly in $\partial\Omega$ and using the asymptotic expression of g, we easily can get that the sign of $\sigma_1 - \lambda_n$ is dictated, for n large enough, by the sign of

$$\int_{\partial\Omega} G(x)\Phi_1^{1+\alpha}.$$

In particular, if this last integral is positive the bifurcation is subcritical and if it is negative the bifurcation is supercritical.

With this in mind, throughout this section we assume:

(H2) For some $\alpha < 1$, for sufficiently large $s > 0$ and $x \in \partial\Omega$, there exists a function G_1 such that

$$\limsup_{|s|\to\infty} \frac{|g(x,s)|}{|s|^\alpha} \le G_1(x), \qquad G_1 \in L^r(\partial\Omega), \quad r > N-1.$$

To evaluate the sign of $\sigma_1 - \lambda_n$, we look at the lower order terms of $g(x,s)$ as $s \to \infty$. Hence, we define, for $\alpha < 1$, the following quantity

$$\underline{\mathbf{G}}_+ \;:=\; \int_{\partial\Omega} \liminf_{s\to+\infty} \frac{sg(x,s)}{|s|^{1+\alpha}}\,\Phi_1^{1+\alpha}, \tag{25}$$

where Φ_1 is the first Steklov eigenfunction as in (3) with $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$. Changing $\liminf$ by $\limsup$ we define the number $\overline{\mathbf{G}}_+$ and considering the limits when $s \to -\infty$ we will have defined $\underline{\mathbf{G}}_-$.

Remark 2.12. (i) Observe that in fact $\underline{\mathbf{G}}_+$ depends on α (and possibly on σ whenever we study bifurcation from any eigenvalue). If we need to stress this dependence, we will write $\underline{\mathbf{G}}_+^{\alpha,\sigma}$, $\overline{\mathbf{G}}_+^{\alpha,\sigma}$.

(ii) Observe that if g satisfies (H1) and $\alpha \ge 1$ then all the functions defined above are identically zero.

(iii) The behavior of the function g for large values of s can be expressed in the following way: for any $\varepsilon > 0$ small enough, we have

$$\left(\liminf_{s\to+\infty} \frac{sg(\cdot,s)}{|s|^{1+\alpha}} - \varepsilon\right) s^\alpha \le g(x,s) \le \left(\limsup_{s\to+\infty} \frac{sg(\cdot,s)}{|s|^{1+\alpha}} + \varepsilon\right) s^\alpha,$$

for $s \to +\infty$. Similarly for $s \to -\infty$.

Those numbers $\underline{\mathbf{G}}_+$, $\overline{\mathbf{G}}_+$, $\underline{\mathbf{G}}_-$ and $\overline{\mathbf{G}}_-$ determine the subcritical or supercritical nature of the bifurcation at σ_1, see Theorem 2.14.

Let us consider a technical lemma that will be the key to prove the result. It is basically a restatement of [7, Lemma 4.2] and it is used to determine whether the bifurcation is subcritical or supercritical. Note that this result allows us to compare σ_1 and λ.

Lemma 2.13. *Assume the nonlinearity g satisfies hypothesis* (H1) *and* (H2) *for $\sigma = \sigma_1$. Denote by σ_1 the first Steklov eigenvalue and by Φ_1 the first positive eigenfunction with $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$ as in (3).*

Consider (λ_n, u_n), a sequence of solutions of (2) such that $\lambda_n \to \sigma_1$ and $\|u_n\|_{L^\infty(\partial\Omega)} \to \infty$. Then, if $u_n > 0$ we have

$$\begin{aligned}\frac{\underline{\mathbf{G}}_+}{\int_{\partial\Omega} {\Phi_1}^2} \quad &\leq \quad \frac{1}{\int_{\partial\Omega} {\Phi_1}^2} \liminf_{n\to\infty} \int_{\partial\Omega} \frac{u_n g(\cdot, u_n)}{|u_n|^{1+\alpha}} \Phi_1^{1+\alpha} \leq \liminf_{n\to\infty} \frac{\sigma_1 - \lambda_n}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \\ &\leq \quad \limsup_{n\to\infty} \frac{\sigma_1 - \lambda_n}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \leq \frac{1}{\int_{\partial\Omega} {\Phi_1}^2} \limsup_{n\to\infty} \int_{\partial\Omega} \frac{u_n g(\cdot, u_n)}{|u_n|^{1+\alpha}} \Phi_1^{1+\alpha} \leq \frac{\overline{\mathbf{G}}_+}{\int_{\partial\Omega} {\Phi_1}^2}.\end{aligned} \tag{26}$$

A similar statement is obtained for the case $u_n < 0$, just changing $\underline{\mathbf{G}}_+$ by $\underline{\mathbf{G}}_-$ and $\overline{\mathbf{G}}_+$ by $\overline{\mathbf{G}}_-$.

Proof. Let us show (i). The other case follows a similar proof. So let us consider a family of solutions u_n of (2) for $\lambda = \lambda_n$ with $\lambda_n \to \sigma_1$ and $0 < u_n \to \infty$. Multiplying equation (2) by Φ_1 and integrating by parts, we get

$$(\sigma_1 - \lambda_n) \int_{\partial\Omega} u_n \Phi_1 = \int_{\partial\Omega} g(x, u_n)\Phi_1. \tag{27}$$

But,

$$\int_{\partial\Omega} g(x, u_n)\Phi_1 = \|u_n\|^\alpha_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \frac{g(x, u_n)}{u_n^\alpha} \left(\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}}\right)^\alpha \Phi_1.$$

Nevertheless, from Fatou's Lemma,

$$\begin{aligned}\liminf_{n\to\infty} \quad &\int_{\partial\Omega} \frac{g(x, u_n)}{u_n^\alpha} \left(\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}}\right)^\alpha \Phi_1 \\ &\geq \int_{\partial\Omega} \liminf_{n\to\infty} \left[\frac{g(x, u_n)}{u_n^\alpha} \left(\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}}\right)^\alpha \Phi_1\right] \\ &\geq \underline{\mathbf{G}}_+,\end{aligned} \tag{28}$$

where we have used the definition of $\underline{\mathbf{G}}_+$, that $\Phi_1 > 0$ for all x on $\partial\Omega$, and the fact that $\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}} \to \Phi_1$ uniformly in $\partial\Omega$ (see Corollary 2.9).

Dividing in (27) by $\|u_n\|_{L^\infty(\partial\Omega)}$ and passing to the limit we obtain the first inequality of (26). The second inequality is trivial and the third is obtained in a similar manner as the first one. ☑

Now, with respect to bifurcations from the first eigenvalue we can prove the following theorem.

Theorem 2.14 (Bifurcation from the first eigenvalue)**.** *Assume the nonlinearity g satisfies hypothesis* (H1) *and* (H2) *for $\sigma = \sigma_1$. Denote by σ_1 the first Steklov eigenvalue and by Φ_1 the first positive eigenfunction with $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$. Then,*

(i) (Subcritical bifurcations). *If*

$$\underline{\mathbf{G}}_+ > 0 \qquad \big(\textit{respectively} \qquad \underline{\mathbf{G}}_- > 0\big) \tag{29}$$

the bifurcation from infinity of positive (resp. negative) solutions at $\lambda = \sigma_1$ is subcritical, i.e., $\lambda < \sigma_1$ for every positive (resp. negative) solution (λ, v) of (2) with $(\lambda, \|v\|)$ in a neighborhood of (σ_1, ∞).

(ii) (Supercritical bifurcations). *If*

$$\overline{\mathbf{G}}_+ < 0 \qquad \big(respectively \qquad \overline{\mathbf{G}}_- < 0\big) \tag{30}$$

the bifurcation from infinity of positive (resp. negative) solutions at $\lambda = \sigma_1$ is supercritical, i.e., $\lambda > \sigma_1$ for every positive (resp. negative) solution (λ, v) of (2) with $(\lambda, \|v\|)$ in a neighborhood of (σ_1, ∞).

Proof. The proof of this Theorem follows directly from Lemma 2.13. Observe that condition (29) impose a definite sign of $\sigma_1 - \lambda_n$ in (26). ☑

As an example of this result we have

Corollary 2.15. *(i) Assume the nonlinearity satisfies $g(x, s) \approx a|s|^\alpha$ as $s \to +\infty$ for some $\alpha < 1$. Then, if $a > 0$ all bifurcations of positive solutions are subcritical, while if $a < 0$ all bifurcations of positive solutions are supercritical.*

(ii) Assume the nonlinearity satisfies $g(x, s) \approx a|s|^\alpha$ as $s \to -\infty$ for some $\alpha < 1$. Then, if $a > 0$ all bifurcations of negative solutions are supercritical, while if $a > 0$ all bifurcations of negative solutions are subcritical.

We consider now the general case, that is, u_n are solutions of (2) for a sequence λ_n with $\lambda_n \to \sigma$ and $\|u_n\|_{L^\infty(\partial\Omega)} \to \infty$. Then, from Proposition 2.8 we have that λ is an eigenvalue and, up to a subsequence, $u_n/\|u_n\|_{L^\infty(\partial\Omega)} \to \Phi$ uniformly for some eigenfunction Φ associated to the eigenvalue σ and with $\|\Phi\|_{L^\infty(\partial\Omega)} = 1$.

We have the following

Theorem 2.16 (Bifurcation from a general eigenvalue)**.** *Let σ be an Steklov eigenvalue for which a bifurcation from infinity of* (2) *occurs at $\lambda = \sigma$. Assume the nonlinearity g satisfies hypothesis* (H1) *and* (H2)*. Then,*

(i) (Subcritical bifurcation). *Assume that $-1 \le \alpha < 1$. For some eigenfunction Φ associated to the eigenvalue σ, let us define the following quantity*

$$\underline{\mathbf{G}}^\sigma := \int_{\partial\Omega} \liminf_{s\to+\infty} \frac{sg(x,s)}{|s|^{1+\alpha}}\ |\Phi^+|^{1+\alpha} + \int_{\partial\Omega} \liminf_{s\to-\infty} \frac{sg(x,s)}{|s|^{1+\alpha}}\ |\Phi^-|^{1+\alpha}, \tag{31}$$

where $\underline{\mathbf{G}}^\sigma = \underline{\mathbf{G}}^\sigma(\Phi)$.

If $\underline{\mathbf{G}}^\sigma > 0$ for any eigenfunction Φ associated to the eigenvalue σ, then the bifurcation from infinity of solutions at $\lambda = \sigma$ is subcritical, i.e., $\lambda < \sigma$ for every solution (λ, v) of (2) *with $(\lambda, \|v\|)$ in a neighborhood of (σ, ∞)*

(ii) (Supercritical bifurcation). *Assume that $-1 \le \alpha < 1$. For some eigenfunction Φ associated to the eigenvalue σ, let us define the following quantity*

$$\overline{\mathbf{G}}^\sigma := \int_{\partial\Omega} \limsup_{s\to+\infty} \frac{sg(x,s)}{|s|^{1+\alpha}}\ |\Phi^+|^{1+\alpha} + \int_{\partial\Omega} \limsup_{s\to-\infty} \frac{sg(x,s)}{|s|^{1+\alpha}}\ |\Phi^-|^{1+\alpha}. \tag{32}$$

If $\overline{\mathbf{G}}^{\sigma} < 0$ for any eigenfunction Φ associated to the eigenvalue σ, then, the bifurcation from infinity of solutions at $\lambda = \sigma$ is supercritical, i.e. $\lambda > \sigma$ for every solution (λ, v) of (2) with $(\lambda, \|v\|)$ in a neighborhood of (σ, ∞)

Proof. We will show the first case. The supercritical case is proved in a similar way.

As in the proof of Theorem 2.14, we need to study the sign of

$$\int_{\partial\Omega} g(x, u_n)\Phi.$$

But, if we denote by $\partial\Omega^+ = \{x \in \partial\Omega : \Phi(x) > 0\}$ and by $\partial\Omega^- = \{x \in \partial\Omega : \Phi(x) < 0\}$, we have

$$\begin{aligned} \int_{\partial\Omega} g(x,u)\Phi &= \int_{\partial\Omega^+} g(x,u)\Phi^+ - \int_{\partial\Omega^-} g(x,u)|\Phi^-| \\ &= \|u\|^\alpha \int_{\partial\Omega^+} \frac{g(x,u)}{(1+|u|)^\alpha}\Phi^+ \left(\frac{1}{\|u\|} + \frac{|u|}{\|u\|}\right)^\alpha \\ &\quad -\|u\|^\alpha \int_{\partial\Omega^-} \frac{g(x,u)}{(1+|u|)^\alpha}|\Phi^-| \left(\frac{1}{\|u\|} + \frac{|u|}{\|u\|}\right)^\alpha . \end{aligned} \tag{33}$$

Observe that, for any $\alpha \geq -1$,

$$\Phi^+ \left(\frac{1}{\|u_n\|} + \frac{|u_n|}{\|u_n\|}\right)^\alpha \to |\Phi^+|^{1+\alpha} \quad \text{in } C(\partial\Omega^+) \qquad \text{as } n \to \infty. \tag{34}$$

Now, passing to the limit in (33), using (34), hypothesis (31) and the Fatou Lemma we conclude the proof. ☑

2.4. A one dimensional example

Now we consider the onedimensional version of (2), where most computations can be made explicit.

Observe that equation (2) in the one dimensional domain $\Omega = (0, 1)$ reads

$$\left\{ \begin{array}{rcll} -u_{xx} + u &=& 0, & \text{in } (0,1), \\ -u_x(0) &=& \lambda u + g(0, u(0)). & \\ u_x(1) &=& \lambda u + g(1, u(1)). & \end{array} \right. \tag{35}$$

The general solution of the differential equation is $u(x) = ae^x + be^{-x}$, and therefore the nonlinear boundary conditions provide two nonlinear equations. in terms of two constants a and b. The function $u = ae^x + be^{-x}$ is a solution if (λ, a, b) satisfy

$$\begin{pmatrix} -(1+\lambda) & (1-\lambda) \\ (1-\lambda)e & -(1+\lambda)e^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} g(0, a+b) \\ g(1, ae + be^{-1}). \end{pmatrix} \tag{36}$$

In this case we only have two Steklov eigenvalues,

$$\sigma_1 = \frac{e-1}{e+1} < \sigma_2 = \frac{1}{\sigma_1} = \frac{e+1}{e-1}.$$

Choosing $g(x,u) = g(u)$ and restricting the analysis to symmetric solutions $u_r(x) = r(e^x + e^{1-x})$, with $r \in \mathbb{R}$, it is easy to prove that $u_r(x)$ is a solution if and only if λ satisfies

$$\lambda(r) = \sigma_1 - \frac{g(r(e+1))}{r(e+1)}, \qquad r > 0. \tag{37}$$

Therefore, whenever $g(u) = o(u)$ at infinity, there is an unbounded branch of solutions $(\lambda(r), u_r) \to (\sigma_1, \infty)$ as $r \to \infty$.

3. The anti-maximum principle for the Steklov problem

Let us consider the following nonhomogeneous linear Steklov problem

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda u + g(x), & \text{on } \partial\Omega. \end{cases} \tag{38}$$

We first show an anti-maximum principle for this problem, Theorem 3.1 (see [13], [3] for the case where the nonlinear term is in Ω). As usual, we denote by σ_1 the first Steklov eigenvalue and by Φ_1 its positive eigenfunction.

Secondly, let us we consider the linear nonhomogeneus problem

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + b(x)u = \lambda u + g(x), & \text{on } \partial\Omega. \end{cases} \tag{39}$$

We will show that there exists a small $\delta > 0$ such that the antimaximum principle holds in $\mu_1(b) < \lambda < \mu_1(b) + \delta$, where $\mu_1(b)$ is the first Steklov eigenvalue associated to (39) (that is, the smallest λ for which there exists a solution of (39) with $g \equiv 0$). The parameter δ can be chosen uniformly for all potentials $b(x)$ lying in a small neighborhood of a given fixed potential $b_0(x)$ and also uniformly in $g(x)$ in certain sense (see Theorem 3.4 below for more details).

In this Section, we show first an antimaximum principle (see Theorem 3.1) and a uniform antimaximum principle with a varying potential (see Theorem 3.4). Also, we state a uniform condition allowing that the antimaximum principle hods (see Corollary 3.5). We also include several results related to the spectral behavior when the potential at the boundary is perturbed. This Section contains a proof of the uniform antimaximum principle and also several technical results on the behavior of the Steklov eigenvalues under variations of the potential at the boundary, which are needed in the section to show the uniform antimaximum principle.

Theorem 3.1. *For every $g \in L^r(\partial\Omega)$ with $r > N-1$, there exists $\varepsilon = \varepsilon(g)$ such that*

1. *If $\int_{\partial\Omega} g\Phi_1 > 0$, then every solution (λ, u) of (8) satisfies the following*
   a) $u > 0$ *if* $\sigma_1 - \varepsilon < \lambda < \sigma_1$,
   b) $u < 0$ *if* $\sigma_1 < \lambda < \sigma_1 + \varepsilon$.

2. *If* $\int_{\partial\Omega} g\Phi_1 = 0$*, then every solution* (λ, u) *of (8) with* $\lambda \neq \sigma_1$ *changes sign on* $\partial\Omega$ *and consequently in* Ω.

Proof. Assume $\int_{\partial\Omega} g\Phi_1 > 0$. The Fredholm Alternative states that the linear problem (8) does not have solution if $\lambda = \sigma_1$ and has a unique solution if $\lambda \notin \sigma(S)$. Moreover from Theorem 2.10, $\lambda = \sigma_1$ is a bifurcation point from infinity, and from Theorem 2.14, the bifurcation from infinity of positive solutions is subcritical, i.e. there exists an $\varepsilon = \varepsilon(g)$ such that for all (λ, u) solving (8) with $\lambda \to \sigma_1$, $\|u\| \approx \infty$ and $u > 0$, then $\sigma_1 - \varepsilon < \lambda < \sigma_1$.

Moreover, by the same theorem, the bifurcation from infinity of negative solutions is supercritical, i.e. there exists an $\varepsilon = \varepsilon(g)$ such that for all (λ, u) solving (8) with $\lambda \to \sigma_1$, $\|u\| \approx \infty$ and $u < 0$, then $\sigma_1 < \lambda < \sigma_1 + \varepsilon$.

Assume now that $\int_{\partial\Omega} g\Phi_1 = 0$. Multiplying equation (8) with $\lambda \neq \sigma_1$, by Φ_1 and integrating by parts, we obtain that $\int_{\partial\Omega} u\Phi_1 = 0$. Since $\Phi_1 > 0$, u has to change sign in $\partial\Omega$ and the proof is concluded. ☑

3.5. A uniform antimaximum principle

Let us consider a family of nonhomogeneous linear Steklov problems containing a potential at the boundary of the form $b_0(x) + \eta(x)$, where $b_0(\cdot) \in L^r(\partial\Omega)$ is a fixed potential and $\eta(\cdot) \in L^r(\partial\Omega)$ will be small in $L^r(\partial\Omega)$ with $r > N - 1$, that is,

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + [b_0(x) + \eta(x))]u = \lambda u + g(x), & \text{on } \partial\Omega. \end{cases} \tag{40}$$

For the analysis in this section, we need to consider several eigenvalue problems. If $b \in L^r(\partial\Omega)$, $r > N - 1$, we denote by $\mu_1(b)$ and $\varphi_1 = \varphi_1(b) > 0$ the first Steklov eigenvalue and eigenfunction of the problem

$$\begin{cases} -\Delta\varphi_1 + \varphi_1 = 0, & \text{in } \Omega, \\ \dfrac{\partial\varphi_1}{\partial n} + b(x)\varphi_1 = \mu_1(b)\varphi_1, & \text{on } \partial\Omega. \end{cases} \tag{41}$$

Also, we will denote by $\Lambda_1(b)$ and $\xi_1 = \xi_1(b) > 0$ the first eigenvalue and eigenfunction, respectively, of the following problem

$$\begin{cases} -\Delta\xi_1 + \xi_1 = \Lambda_1\xi_1, & \text{in } \Omega, \\ \dfrac{\partial\xi_1}{\partial n} + b(x)\xi_1 = 0, & \text{on } \partial\Omega. \end{cases} \tag{42}$$

From maximum principles, it is well known that if $b_1 \leq b_2$, $b_1 \neq b_2$, then $\mu_1(b_1) < \mu_1(b_2)$ and $\Lambda_1(b_1) < \Lambda_1(b_2)$.

Note also that for both (41) and (42), the first eigenvalue is simple and is the only one with a positive associated eigenfunction.

We will refer to Λ_1 in (42) as the *interior* eigenvalue, to distinguish it clearly from the *boundary* Steklov eigenvalue, μ_1 in (41). We will keep this notation on eigenvalues and eigenfunctions throughout the section. Also, the first eigenfunction will be normalized in $L^\infty(\partial\Omega)$, unless otherwise stated.

We will denote by $\mu_i^\eta := \mu_i(b_0+\eta)$ and $\varphi_i^\eta := \varphi_i(b_0+\eta)$, $i = 1, 2, \ldots$, the Steklov eigenvalues and eigenfunctions of the problem

$$\left\{\begin{array}{rcll} -\Delta\varphi + \varphi &=& 0, & \text{in } \Omega, \\ \dfrac{\partial\varphi}{\partial n} + [b_0(x)+\eta(x)]\varphi &=& \mu\varphi, & \text{on } \partial\Omega. \end{array}\right. \tag{43}$$

We will denote by μ_i^0 and φ_i^0, $i = 1, 2, \ldots$, the Steklov eigenvalue and eigenfunction of the problem

$$\left\{\begin{array}{rcll} -\Delta\varphi_i^0 + \varphi_i^0 &=& 0, & \text{in } \Omega, \\ \dfrac{\partial\varphi_i^0}{\partial n} + b_0(x)\varphi_i^0 &=& \mu_i^0\varphi_i^0, & \text{on } \partial\Omega. \end{array}\right. \tag{44}$$

We start with a result on the behavior of the solutions of (40) and of the spectra of (43) (see [8, Proposition A.1] for a proof).

Proposition 3.2. *Let us consider a family of potentials $\eta_n \in L^r(\partial\Omega)$ for some $r > N-1$, satisfying $\eta_n \rightharpoonup 0$, weakly in $L^r(\partial\Omega)$. Denote by $S_\eta : L^r(\partial\Omega) \to L^r(\partial\Omega)$, the solution operator of (40) with $\lambda = 0$, that is $S_\eta(g) = \gamma(u)$, where u is the solution of (40) with $\lambda = 0$ and $\gamma(\cdot)$ is the trace operator. Then, there exists a large enough constant $a > 0$ such that S_a and $S_{a+\eta_n}$ are well defined and*

$$\|S_{a+\eta_n} - S_a\|_{\mathcal{L}(L^r(\partial\Omega), L^r(\partial\Omega))} \to 0, \text{ as } n \to +\infty. \tag{45}$$

Moreover, we have the convergence of eigenvalues and eigenfunctions, that is $\mu_i^{\eta_n} \to \mu_i^0$ as $n \to +\infty$ for all $i = 1, 2, ..$, and in particular

$$\varphi_1^{\eta_n} \to \varphi_1^0, \text{ in } H^1(\Omega), C^\alpha(\bar\Omega) \tag{46}$$

for some $\alpha > 0$.

In a very similar way we have the following proposition.

Proposition 3.3. *Let us consider a family of potentials $\eta_n \in L^r(\partial\Omega)$ for some $r > N-1$, satisfying $\eta_n \rightharpoonup 0$, weakly in $L^r(\partial\Omega)$. Denote by $T_{\eta,c} : L^2(\Omega) \to L^2(\Omega)$ the solution operator of*

$$\left\{\begin{array}{rcll} -\Delta u + cu &=& f, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + [b_0(x)+\eta(x))]u &=& 0, & \text{on } \partial\Omega, \end{array}\right. \tag{47}$$

that is $T_{\eta,c}(f) = u$, where u is the solution of (47). Then, there exists a large enough constant $c > 0$ such that $T_{\eta_n,c}$ and $T_{0,c}$ are well defined and

$$\|T_{\eta_n,c} - T_{0,c}\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} \to 0, \text{ as } n \to +\infty. \tag{48}$$

Moreover, we have the convergence of eigenvalues and eigenfunctions, that is, with the notations or (42), $\Lambda_i(b+\eta_n) \to \Lambda_i(b)$ as $n \to +\infty$ for all $i = 1, 2, ..$, and similarly for the eigenfunctions.

Proof. The proof follows the same ideas as the proof of Proposition 3.2. To show (48) we pass to the limit in the weak formulation of (47) and use elliptic regularity theory to show that the convergence is in stronger norms. The convergence of the eigenvalues and eigenfunctions follows from (48) (see again [24]). ☑

Now, we want to analyze the behavior of the solutions of (40) with λ varying in a neighborhood of μ_1^0 and assuming that $\|\eta\|_{L^r(\partial\Omega)}$ is small. As a matter of fact, we can show the following theorem.

Theorem 3.4. *There exist three constants $\eta_0, d_0, M > 0$ such that for every function $\eta \in L^r(\partial\Omega)$ with $\|\eta\|_{L^r(\partial\Omega)} \leq \eta_0$, and every function $g \in L^r(\partial\Omega)$ with $r > N-1$, and $\int_{\partial\Omega} g\,\varphi_1^\eta > 0$, we have*

(i) *if* $\lambda \in \left(\mu_1^\eta, \mu_1^\eta + M\, \dfrac{\int_{\partial\Omega} g\,\varphi_1^\eta}{\|g\|_{L^r(\partial\Omega)}}\right) \cap I$ *then* $u < 0$,

(ii) *if* $\lambda \in \left(\mu_1^\eta - M\, \dfrac{\int_{\partial\Omega} g\,\varphi_1^\eta}{\|g\|_{L^r(\partial\Omega)}}\,, \mu_1^\eta\right) \cap I$, *then* $u > 0$,

where $I = [\mu_1^0 - d_0, \mu_1^0 + d_0]$ and u is the solution of (40).

Proof. For each $\eta \in L^r(\partial\Omega)$ fixed, we consider

$$L^r(\partial\Omega) = \operatorname{span}[\varphi_1^\eta] \oplus \operatorname{span}[\varphi_1^\eta]^\perp, \tag{49}$$

where

$$\operatorname{span}[\varphi_1^\eta]^\perp := \left\{ u \in L^r(\partial\Omega) : \int_{\partial\Omega} u\,\varphi_1^\eta = 0 \right\}, \tag{50}$$

and therefore, for every $g \in L^r(\partial\Omega)$ with $r > N-1$ there exists a unique decomposition

$$g = a_0(\eta)\varphi_1^\eta + g_1^\eta, \quad \text{where} \quad a_0(\eta) := \frac{\int_{\partial\Omega} g\,\varphi_1^\eta}{\int_{\partial\Omega} |\varphi_1^\eta|^2}, \quad \text{and} \quad \int_{\partial\Omega} g_1^\eta \varphi_1^\eta = 0. \tag{51}$$

The well known Fredholm Alternative states that the linear problem (40) for $\lambda \in \mathbb{R}$ does not have solution if $\lambda \in \{\mu_i^\eta\}_{i=1}^\infty$ and has a unique solution if $\lambda \neq \mu_i^\eta$, for all $i = 1, 2 \ldots$. The solution u in the latter case has a unique decomposition

$$u = \frac{a_0(\eta)}{\mu_1^\eta - \lambda}\, \varphi_1^\eta + u_1, \quad \text{with} \quad \int_{\partial\Omega} u_1 \varphi_1^\eta = 0, \tag{52}$$

where $a_0(\eta)$ is defined in (51) and $u_1 = u_1(\eta, \lambda)$ solves the following problem

$$\left\{ \begin{array}{rcll} -\Delta u_1 + u_1 &=& 0, & \text{in } \Omega, \\ \dfrac{\partial u_1}{\partial n} + [b_0(x) + \eta(x)]\, u_1 &=& \lambda u_1 + g_1^\eta, & \text{on } \partial\Omega. \end{array} \right. \tag{53}$$

Moreover, by the decomposition of g, see (51), $u_1 \in \operatorname{span}[\varphi_1^\eta]^\perp$. By hypothesis and from the Fredholm Alternative, it is already known that the linear problem (53) has a unique solution u_1 in $\operatorname{span}[\varphi_1^\eta]^\perp$.

From the continuous dependence of the Steklov eigenvalues with respect to the potential given by Proposition 3.2, we know that we have that $\mu_i^\eta \to \mu_i^0$ for all $i = 1, 2. \dots$ and

$$\varphi_1^\eta \to \varphi_1^0 \quad \text{in} \quad C^\alpha(\overline{\Omega}) \quad \text{for some} \quad 0 < \alpha < 1, \quad \text{as } \|\eta\|_{L^r(\partial\Omega)} \to 0, \tag{54}$$

which implies that we can choose $\tilde{\eta}_0 > 0$ small such that

$$\min_{x\in\bar{\Omega}} \frac{\varphi_1^\eta(x)}{\int_{\partial\Omega} |\varphi_1^\eta|^2} \geq \frac{1}{2} \min_{x\in\bar{\Omega}} \frac{\varphi_1^0(x)}{\int_{\partial\Omega} |\varphi_1^0|^2} > 0, \text{ for } \|\eta\|_{L^r(\partial\Omega)} \leq \tilde{\eta}_0. \tag{55}$$

Let $d_0 = (\mu_2^0 - \mu_1^0)/2 > 0$ and let us consider now $0 < \eta_0 \leq \tilde{\eta}_0$ small enough with the property that for each $\eta \in L^r(\partial\Omega)$ with $\|\eta\|_{L^r(\partial\Omega)} \leq \eta_0$, we have $[\mu_1^0 - d_0, \mu_1^0 + d_0] \cap \{\eta\}_{i=1}^\infty = \mu_1^\eta$.

Let us define the set

$$E = \{(\lambda, \eta) \in [\mu_1^0 - d_0, \mu_1^0 + d_0] \times L^r(\partial\Omega) \qquad \text{with} \quad \|\eta\|_{L^r(\partial\Omega)} \leq \eta_0 \qquad \text{and} \quad \lambda \neq \mu_1^\eta\}.$$

We will next prove that for a fixed $g \in L^r(\partial\Omega)$, $u_1 = u_1(\lambda, \eta)$ is uniformly bounded for any $(\lambda, \eta) \in E$.

Let us argue by contradiction. If this is not the case, then there exists a sequence $(\lambda_n, \eta_n) \in E$ such that $\|u_1(\lambda_n, \eta_n)\|_{L^\infty(\partial\Omega)} \to \infty$. Taking another subsequence if necessary, we may assume that there exists $\eta \in L^r(\partial\Omega)$ such that $\eta_n \rightharpoonup \eta$, weakly in $L^r(\partial\Omega)$. Applying Proposition 3.2 we get that $\mu_1^{\eta_n} \to \mu_1^\eta$ and $\varphi_1^{\eta_n} \to \varphi_1^\eta$ in $C^\alpha(\bar{\Omega})$. Arguing as in [7, Proposition 3.1], we get that necessarily this sequence must satisfy $\lambda_n \to \mu_1^\eta$ and, at least for another subsequence, that we denote the same,

$$\left\| \frac{u_1(\lambda_n, \eta_n)}{\|u_1(\lambda_n, \eta_n)\|_{L^\infty(\partial\Omega)}} - \varphi_1^\eta \right\|_{L^\infty(\Omega)} \to 0.$$

This is in contradiction with the fact that $u_1(\lambda) \in \text{span}[\varphi_1^{\eta_n}]^\perp$ and the convergence in (54).

Let us now define a family of operators $T(\lambda, \eta) : L^r(\partial\Omega) \to L^\infty(\Omega)$ for $(\lambda, \eta) \in E$, by $T(\lambda, \eta)(g) := u_1(\lambda, \eta)$. From elliptic regularity, $T(\lambda, \eta)$ is continuous. Moreover $\|T(\lambda, \eta)(g)\|_{L^\infty(\Omega)} \leq C(g)$ for all $(\lambda, \eta) \in E$. Therefore, applying the uniform boundedness principle, there exists a constant C_1 such that

$$\|u_1(\lambda, \eta)\|_{L^\infty(\partial\Omega)} \leq C_1 \|g\|_{L^r(\partial\Omega)} \quad \text{for any } (\lambda, \eta) \in E. \tag{56}$$

Consider the case $\mu_1^\eta < \lambda$. From (52) and (56) we have that for $(\lambda, \eta) \in E$ we have

$$u \leq \frac{a_0(\eta)}{\mu_1^\eta - \lambda} \, \varphi_1^\eta + C_1 \|g\|_{L^r}.$$

From here, if we define $C(\eta) := \min_{x\in\overline{\Omega}} \varphi_1^\eta(x) / \left(C_1 \int_{\partial\Omega} |\varphi_1^\eta|^2\right)$, we obtain that for $(\lambda, \eta) \in E$, if $0 < \lambda - \mu_1^\lambda < C(\eta) \, \dfrac{\int_{\partial\Omega} g \, \varphi_1^\eta}{\|g\|_{L^r}}$, then $u < 0$.

Now, taking into account (55) we have

$$C(\eta) \geq \frac{1}{2C_1} \min_{x\in\bar{\Omega}} \frac{\varphi_i^0(x)}{\int_{\partial\Omega} |\varphi_1^0|^2} := M > 0, \text{ for } \|\eta\|_{L^r(\partial\Omega)} \leq \eta_0,$$

from where (i) follows. The other inequality is obtained in a similar way. ☑

Let us finally consider a family of nonhomogeneous linear Steklov problems with a variable nonhomogeneous term at the boundary g depending on the parameter λ

$$\left\{ \begin{array}{rcll} -\Delta u + u &=& 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + [b_0(x) + \eta(\lambda, x)]u &=& \lambda u + g(x), & \text{on } \partial\Omega, \end{array} \right. \tag{57}$$

where $g(\lambda, \cdot) \in L^r(\partial\Omega)$ and $b_0 + \eta(\lambda, \cdot) \in L^r(\partial\Omega)$. We will also assume that

$$\|\eta(\lambda, \cdot)\|_{L^r(\partial\Omega)} \to 0 \qquad \text{as} \quad \lambda \to \mu_1^0.$$

Corollary 3.5. *Assume that the following hypothesis holds*

$$\|\eta(\lambda, \cdot)\|_{L^r(\partial\Omega)} \to 0, \qquad \textit{as} \quad \lambda \to \mu_1^0. \tag{58}$$

Assume also that $\|g(\cdot)\|_{L^r(\partial\Omega)} \neq 0$ *for all* $\lambda \in [\mu_1^0 - \delta_0, \mu_1^0 + \delta_0]$ *for some* $\delta_0 > 0$, *and that*

$$\liminf_{\lambda \to d_0} \int_{\partial\Omega} \frac{g(\cdot)\, \varphi_1^0}{\|g(\cdot)\|_{L^r(\partial\Omega)}} > 0. \tag{59}$$

Then, there exist constants $\delta, \tilde{M} > 0$ *such that*

(i) *if* $\lambda \in \left(\mu_1^{\eta(\lambda)}, \mu_1^{\eta(\lambda)} + \tilde{M}\right) \cap I$ *then* $u < 0$,

(ii) *if* $\lambda \in \left(\mu_1^{\eta(\lambda)} - \tilde{M}, \mu_1^{\eta(\lambda)}\right) \cap I$, *then* $u > 0$.

where $I = [\mu_1^0 - \delta, \mu_1^0 + \delta]$ *and* u *is the solution of (57) .*

Proof. Define $\tilde{g}(\cdot) = g(\cdot)/\|g(\cdot)\|_{L^r(\partial\Omega)}$ and $\tilde{u} = u/\|g(\cdot)\|_{L^r(\partial\Omega)}$ so that $\tilde{u}$ satisfies

$$\left\{ \begin{array}{rcll} -\Delta \tilde{u} + \tilde{u} &=& 0, & \text{in } \Omega, \\ \dfrac{\partial \tilde{u}}{\partial n} + [b_0(x) + \eta(\lambda, x)]\tilde{u} &=& \lambda \tilde{u} + \tilde{g}(x), & \text{on } \partial\Omega. \end{array} \right. \tag{60}$$

From the convergence of $\varphi_1^{\eta(\lambda)}$ to φ_1^0 stated in (54) and from (59) we get

$$\liminf_{\lambda\to\mu_1^0} \int_{\partial\Omega} \tilde{g}(\cdot)\, \varphi_1^{\eta(\lambda)} \geq \liminf_{\lambda\to\mu_1^0} \int_{\partial\Omega} \tilde{g}(\cdot)\, [\varphi_1^{\eta(\lambda)} - \varphi_1^0] + \liminf_{\lambda\to\mu_1^0} \int_{\partial\Omega} \tilde{g}(\cdot)\, \varphi_1^0 > 0,$$

from where we obtain that there exists $a_0 > 0$ and $\delta > 0$ such that for $\lambda \in [\mu_1^0 - \delta, \mu_1^0 + \delta]$ we have

$$\int_{\partial\Omega} \tilde{g}(\cdot)\, \varphi_1(\lambda, \cdot) \geq a_0, \quad \lambda \in [\mu_1^0 - \delta, \mu_1^0 + \delta].$$

Now the result is a consequence of the theorem above. ☑

4. Stability

In this Section we consider the nonlinear parabolic equation with nonlinear boundary conditions (1) and analyze the behavior and stability properties of the equilibrium solutions as well as some features of the global dynamics. The equilibria are the solutions of the elliptic problem with nonlinear boundary conditions (2).

In this Section, we characterize the stability of equilibria and analyze several features of the bifurcating branches. The stability is characterized in terms of two ordered numbers which depend on the asymptotic behavior of g and g_u at infinity. Whenever both numbers are positive (resp. negative), any positive solution contained in the unbounded branch is stable, subcritical and unique for each λ (resp. unstable and supercritical) in a neighborhood of bifurcation point at infinity.

First, we give conditions, which involve a more detailed knowledge of the behavior of the nonlinear term as $|u| \to \infty$, which imply that the unbounded branch of positive equilibria is subcritical, unique and stable (see Theorem 4.5). In an almost exact complementary situation, we also show that the unbounded branch of positive equilibria is supercritical, unique and unstable (see Theorem 4.6).

Let us mention that all these results, which are described in the introduction for positive solutions, have analogous statements for the negative branch of solutions.

The section is organized as follows. In Subsection 4.6 we make precise the hypotheses on the nonlinearity and collect some notations and known results. We also give a more precise description of some of the results in the section. Subsection 4.7 contains our stability results for the solutions of (2).

4.6. Preliminaries and description of the results

In this Subsection we review the setting and results from Section 2, which we take as a starting point for our analysis. We also describe in a more technical and detailed way our results.

With respect to the nonlinearity g in (1) and (2), we assume hypotheses (H1), (H2) and also the following hypothesis

(H3) The nonlinearity $g_s(x,s)$ is differentiable in s and its partial derivative $g_s(\cdot,\cdot) \in C(\partial\Omega \times \mathbb{R})$, where $g_s := \dfrac{\partial g}{\partial s}$, and there exist $F_1 \in L^r(\partial\Omega)$, with $r > N-1$, and $\rho < 1$ such that

$$\frac{|g(x,s) - s g_s(x,s)|}{|s|^\rho} \leq F_1(x), \qquad \text{as } \lambda \to \sigma_1 \tag{61}$$

for $x \in \partial\Omega$ and $s \gg 1$ sufficiently large.

Elliptic regularity results and bootstrap arguments imply that solving (2) in, say $H^1(\Omega)$, is equivalent to solving the problem in a more regular space like Hölder spaces (see Proposition 2.3). Hence, we may consider the solution pair (λ, u) of (2) in $\mathbb{R} \times C(\bar{\Omega})$. Since g is sublinear at infinity, the linear part of the boundary condition of (2) is the dominant term for u large enough. Hence, it is expected that large solutions of (2) can only exist, due to parametric resonance at the boundary, that is, when λ is near a Steklov eigenvalue (see (3). This was already proved in Theorem 2.10 (see also [7, Proposition 3.1, Theorem 3.3]), at an eigenvalue of odd multiplicity. In particular this holds at σ_1, which is the case we consider in this Section. These results were obtained by showing that bifurcation from infinity occurs at such eigenvalues (cf. [31]). Furthermore we have (6) and (7).

To elucidate whether or not the unbounded branch of solutions of (2) is subcritical or supercritical, the quantities $\underline{\mathbf{G}}_+, \overline{\mathbf{G}}_+$ (see (25)), which measure the asymptotic behavior of the nonlinear term at infinity, were used. It is shown in Theorem 2.14 that, if $\underline{\mathbf{G}}_+ > 0$, the positive unbounded branch of equilibria is subcritical, while it is supercritical if $\overline{\mathbf{G}}_+ < 0$.

To determine the stability of the solutions u_λ of (2) bifurcating from infinity at the first Steklov eigenvalue, σ_1, one must determine the sign of the first eigenvalue, Λ_1, of the linearized problem

$$\left\{ \begin{array}{rcll} -\Delta\xi + \xi & = & \Lambda\xi, & \text{in } \Omega, \\ \dfrac{\partial \xi}{\partial n} & = & \lambda\xi + g_u(x, u_\lambda)\xi, & \text{on } \partial\Omega, \end{array} \right.$$

where $g_u = \dfrac{\partial g}{\partial u}$, as $\lambda \to \sigma_1$.

This will be obtained in terms of the following quantities, which involve a more detailed account of the asymptotic behavior of the nonlinear term at infinity and as $\lambda \to \sigma_1$:

$$\underline{\mathbf{F}}_+ := \int_{\partial\Omega} \liminf_{s\to+\infty} \frac{sg(\cdot, s) - s^2 g_u(\cdot, s)}{|s|^{1+\rho}} \Phi_1^{1+\rho}, \tag{62}$$

and

$$\overline{\mathbf{F}}_+ := \int_{\partial\Omega} \limsup_{s\to+\infty} \frac{sg(\cdot, s) - s^2 g_u(\cdot, s)}{|s|^{1+\rho}} \Phi_1^{1+\rho},$$

for some $\rho < 1$. In this Section we show that, if $\underline{\mathbf{F}}_+ > 0$, any positive large solution is stable, subcritical and unique for each λ in a neighborhood of σ_1 (see Theorem 4.5). On the other hand, if $\overline{\mathbf{F}}_+ < 0$, any positive large solution is unstable, supercritical and unique in a neighborhood of σ_1 (see Theorem 4.6).

For example, if

$$g(x, s) := a(x)s^\alpha, \qquad \text{for} \quad s \gg 1,$$

and $a(x)$ is such that $\int_{\partial\Omega} a\Phi_1^{1+\alpha} > 0$, then $\underline{\mathbf{F}}_+ > 0$. If, on the contrary, $\int_{\partial\Omega} a\Phi_1^{1+\alpha} < 0$, then $\overline{\mathbf{F}}_+ < 0$.

Remark 4.1. Let us observe that if

$$g(x, s) := a(x)s^\rho, \qquad \text{for} \quad |s| \ll 1,$$

and $a(x)$ is such that $\int_{\partial\Omega} a\Phi_1^{1+\alpha} > 0$, then $\overline{\mathbf{F}}_+ < 0$, and any positive solution bifurcating from zero is unstable, supercritical and unique in a neighborhood of $(\sigma_1, 0)$. If, on the contrary, $\int_{\partial\Omega} a\Phi_1^{1+\alpha} < 0$, then $\underline{\mathbf{F}}_+ < 0$, and any positive solution bifurcating from zero is stable, subcritical and unique for each λ in a neighborhood of $(\sigma_1, 0)$.

4.7. Stability or instability of positive equilibria bifurcating from infinity

We analyze in this Section the stability properties of the branches of solutions of (2) bifurcating from infinity at the first Steklov eigenvalue σ_1.

We sketch now the main argument that will lead to the stability and instability result. Let us denote by $u_\lambda > 0$ a solution of (2) bifurcating from infinity for λ near σ_1. The eigenvalue problem associated to the linearization around u_λ, as an equilibrium of (1), is given by

$$\left\{ \begin{array}{rcll} -\Delta\xi + \xi &=& \Lambda\xi, & \text{in } \Omega, \\ \dfrac{\partial \xi}{\partial n} &=& \lambda\xi + g_u(x, u_\lambda)\xi, & \text{on } \partial\Omega, \end{array} \right. \tag{63}$$

where $g_u = \dfrac{\partial g}{\partial u}$. Thus the stability properties of u_λ are determined by the sign of the first eigenvalue of (63). Following the notations in (42), the eigenvalue can be written as $\Lambda_1 := \Lambda_1(-\lambda - g_u(x, u_\lambda))$.

Let us also consider the auxiliary Steklov eigenvalue problem associated to the linearization around u_λ given by

$$\left\{ \begin{array}{rcll} -\Delta\varphi + \varphi &=& 0, & \text{in } \Omega, \\ \dfrac{\partial \varphi}{\partial n} &=& \mu\varphi + g_u(x, u_\lambda)\varphi, & \text{on } \partial\Omega. \end{array} \right. \tag{64}$$

Observe that with the notations of (41), the first eigenvalue of (64) can be written as $\mu_1 := \mu_1(-g_u(\cdot, u_\lambda))$.

Now we use that for both eigenvalue problems (63), (64) the first eigenvalue is the only one with a positive eigenfunction. This implies that in (64) the first *interior* eigenvalue associated to the boundary potential $b(x) = -\mu_1 - g_u(\lambda, x, u_\lambda)$ satisfies $\Lambda_1(-\mu_1 - g_u(\lambda, x, u_\lambda)) = 0$, while in (63) the first eigenvalue is $\Lambda_1(-\lambda - g_u(\lambda, x, u_\lambda))$. Hence, if we are able to compare μ_1 in (64) with λ, then in (63) we will have that u_λ is stable if $\mu_1 > \lambda$, and unstable if $\mu_1 < \lambda$.

Therefore, we need to figure out a tool to compare μ_1 with λ, as $\lambda \to \sigma_1$. This will be achieved in Lemma 4.4 below. For this we look at the lower order terms of $g(x, s)$ as $\lambda \to \sigma_1$ and $s \to \infty$. Hence, consider, for some $\alpha, \rho < 1$, the quantities $\underline{\mathbf{G}}_+$, and $\underline{\mathbf{F}}_+$ defined by (25) and (62) respectively.

Remark 4.2. Observe that (H2) and (H3) imply that

$$\frac{|g_s(x, s)|}{|s|^{\gamma-1}} \le |s|^{\rho-\gamma} F_1(x) + |s|^{\alpha-\gamma} G_1(x), \qquad \text{as } \lambda \to \sigma_1, \qquad \text{for } s \gg 1,$$

where $\gamma = \max\{\rho, \alpha\} < 1$. Hence,

$$\frac{|g_s(x, s)|}{|s|^{\gamma-1}} \le D_1(x) \qquad \text{with} \quad D_1 \in L^r(\partial\Omega) \qquad \text{with} \quad r > N - 1, \tag{65}$$

for s big enough, $x \in \partial\Omega$ and $\lambda \to \sigma_1$. Therefore, we can also define

$$\underline{\mathbf{D}}_+ \ := \ \int_{\partial\Omega} \liminf_{s \to +\infty} \frac{g_u(x, s)}{|s|^{\gamma-1}} \ \Phi_1^{1+\gamma}, \tag{66}$$

Changing lim inf by lim sup we define the numbers $\overline{\mathbf{G}}_+$, $\overline{\mathbf{F}}_+$, $\overline{\mathbf{D}}_+$, and considering the limits when $s \to -\infty$ we will have defined $\underline{\mathbf{G}}_-$, $\underline{\mathbf{F}}_-$, $\underline{\mathbf{D}}_-$ and $\overline{\mathbf{G}}_-$, $\overline{\mathbf{F}}_-$, $\overline{\mathbf{D}}_-$.

Note that $\underline{\mathbf{G}}_+$, $\overline{\mathbf{G}}_+$, $\underline{\mathbf{G}}_-$ and $\overline{\mathbf{G}}_-$ where used in [7] to determine the subcritical or supercritical nature of the bifurcation at σ_1.

Observe that the difficulty of comparing μ_1 and λ is that, as $\lambda \to \sigma_1$, we have $\mu_1 \to \sigma_1$ as well (see Lemma 4.3 below).

Let us consider now a technical lemma that, joint with Lemma 2.13 will be the key to prove Lemma 4.4. Lemma 2.13 was used to determine whether the bifurcation is subcritical or supercritical. Note that this result allows us to compare σ_1 and λ.

Let us now denote by $u_\lambda > 0$ a solution of (2) bifurcating from infinity. We consider the auxiliary linearized Steklov eigenvalue problem (64) and, with the notations in (41), denote the first eigenvalue by $\mu_1 = \mu_1(-g_u(\cdot, u_\lambda))$ and the first positive eigenfunction by $\varphi_1 = \varphi_1(\lambda, u_\lambda)$, which we assume normalized in $L^\infty(\partial\Omega)$ so that $\|\varphi_1\|_{L^\infty(\partial\Omega)} = 1$.

The next result states sufficient conditions for the convergence of $\mu_1 \to \sigma_1$ and of $\varphi_1 \to \Phi_1$ as $\lambda \to \sigma_1$, and allows to compare μ_1 and σ_1.

Lemma 4.3. *Assume the nonlinearity g satisfies hypotheses* (H1), (H2) *and* (H3).

Then, the first eigenvalue and eigenfunction in (64) satisfy

$$\mu_1(-g_u(\cdot, u_\lambda)) \to \sigma_1 \quad \text{as} \quad \lambda \to \sigma_1, \tag{67}$$

$$\varphi_1(u_\lambda) \to \Phi_1 \quad \text{in } H^1(\Omega) \cap C^\beta(\overline{\Omega}) \quad \text{as} \quad \lambda \to \sigma_1, \tag{68}$$

for some $\beta \in (0, 1)$.

Moreover, for any sequence of solutions of (2), (λ_n, u_n) such that $\lambda_n \to \sigma_1$ and $\|u_n\|_{L^\infty(\partial\Omega)} \to \infty$, setting $\mu_{1,n} = \mu_1(-g_u(\cdot, u_n))$, we have, if $u_n > 0$

$$\frac{\underline{\mathbf{D}}_+}{\int_{\partial\Omega} \Phi_1{}^2} \le \liminf_{n\to\infty} \frac{\sigma_1 - \mu_{1,n}}{\|u_n\|^{\gamma-1}_{L^\infty(\partial\Omega)}} \le \limsup_{n\to\infty} \frac{\sigma_1 - \mu_{1,n}}{\|u_n\|^{\gamma-1}_{L^\infty(\partial\Omega)}} \le \frac{\overline{\mathbf{D}}_+}{\int_{\partial\Omega} \Phi_1{}^2}, \tag{69}$$

where $\gamma = \max\{\rho, \alpha\} < 1$.

A similar statement is obtained for the case $u_n < 0$, just changing $\underline{\mathbf{D}}_+$ by $\underline{\mathbf{D}}_-$ and $\overline{\mathbf{D}}_+$ by $\overline{\mathbf{D}}_-$.

Proof. Note that, using $\gamma < 1$, (65) and (6), in (64) the boundary potential satisfies

$$g_u(\cdot, u_\lambda) = |u_\lambda| \frac{g_u(\cdot, u_\lambda)}{|u_\lambda|^{\gamma-1}} \to 0 \quad \text{in } L^r(\partial\Omega),$$

as $\lambda \to \sigma_1$.

From this, the spectrum of the linear operator also passes to the limit since $r > N - 1$, and then $\varphi_1(\lambda, u_\lambda) \to \Phi_1$ in $H^1(\Omega)$ as $\lambda \to \sigma_1$ (see Proposition 3.3 in Section 3). The elliptic regularity imply now that (68) is satisfied.

On the other hand, if $u_n > 0$, considering equation (64) for the first eigenfunction, multiplying it by Φ_1 and integrating by parts, we get

$$(\sigma_1 - \mu_{1,n}) \int_{\partial\Omega} \varphi_{1,n} \Phi_1 = \int_{\partial\Omega} g_u(\cdot, u_n) \varphi_{1,n} \Phi_1, \tag{70}$$

where $\varphi_{1,n} = \varphi_1(\lambda_n, u_n)$. But,

$$\int_{\partial\Omega} g_u(\cdot, u_n)\varphi_{1,n}\Phi_1 = \|u_n\|^{\gamma-1}_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \frac{g_u(\cdot, u_n)}{|u_n|^{\gamma-1}} \left(\frac{|u_n|}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^{\gamma-1} \varphi_{1,n}\Phi_1,$$

and

$$\begin{aligned}
\liminf_{n\to\infty} \quad & \int_{\partial\Omega} \frac{g_u(\cdot, u_n)}{|u_n|^{\gamma-1}} \left(\frac{|u_n|}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^{\gamma-1} \varphi_{1,n}\Phi_1 \\
\geq \quad & \liminf_{n\to\infty} \int_{\partial\Omega} \frac{g_u(\cdot, u_n)}{|u_n|^{\gamma-1}} \left[\left(\frac{|u_n|}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^{\gamma-1} - \Phi_1^{\gamma-1} \right] \varphi_{1,n}\Phi_1 \\
& + \liminf_{n\to\infty} \int_{\partial\Omega} \frac{g_u(\cdot, u_n)}{|u_n|^{\gamma-1}} \ \varphi_{1,n} \ \Phi_1^{\gamma} \\
\geq \quad & \liminf_{n\to\infty} \int_{\partial\Omega} \frac{g_u(\cdot, u_n)}{|u_n|^{\gamma-1}} \left[\varphi_{1,n} - \Phi_1 \right] \ \Phi_1^{\gamma} \\
& + \int_{\partial\Omega} \liminf_{n\to\infty} \frac{g_u(\cdot, u_n)}{|u_n|^{\gamma-1}} \ \Phi_1^{1+\gamma} \geq \underline{\mathbf{D}}_+,
\end{aligned}$$

where we have used again that $\Phi_1 > 0$ for all x on $\partial\Omega$, (6), (68) and Fatou's Lemma. Dividing in (70) by $\|u_n\|^{\gamma-1}_{L^\infty(\partial\Omega)}$ and passing to the limit we obtain the first inequality of (69). The second inequality is obvious and the third one is obtained similarly to the first one. ☑

We are now in a position to prove the following result, from which stability and instability will be derived. Note that this result allows us to compare λ and μ_1 as $\lambda \to \sigma_1$.

Lemma 4.4. *Assume the nonlinearity g satisfies hypotheses* (H1), (H2) *and* (H3). *Then, for any sequence of solutions of* (2) (λ_n, u_n) *such that $\lambda_n \to \sigma_1$ and $\|u_n\|_{L^\infty(\partial\Omega)} \to \infty$, denoting by $\mu_{1,n} = \mu_1(-g_u(\cdot, u_n))$, the first eigenvalue in (64), we have, if $u_n > 0$*

$$\begin{aligned}
\frac{\underline{\mathbf{F}}_+}{\int_{\partial\Omega} {\Phi_1}^2} \quad & \leq \quad \frac{1}{\int_{\partial\Omega} {\Phi_1}^2} \ \liminf_{n\to\infty} \int_{\partial\Omega} \frac{u_n g(x, u_n) - u_n^2 g_u(x, u_n)}{|u_n|^{1+\rho}} \Phi_1^{1+\rho} \\
& \leq \quad \liminf_{n\to\infty} \frac{\mu_{1,n} - \lambda_n}{\|u_n\|^{\rho-1}_{L^\infty(\partial\Omega)}} \leq \limsup_{n\to\infty} \frac{\mu_{1,n} - \lambda_n}{\|u_n\|^{\rho-1}_{L^\infty(\partial\Omega)}} \\
& \leq \quad \frac{1}{\int_{\partial\Omega} {\Phi_1}^2} \ \limsup_{n\to\infty} \int_{\partial\Omega} \frac{u_n g(x, u_n) - u_n^2 g_u(x, u_n)}{|u_n|^{1+\rho}} \Phi_1^{1+\rho} \leq \frac{\overline{\mathbf{F}}_+}{\int_{\partial\Omega} {\Phi_1}^2}.
\end{aligned}$$

A similar statement is obtained for the case $u_n < 0$, just changing $\underline{\mathbf{F}}_+$ by $\underline{\mathbf{F}}_-$ and $\overline{\mathbf{F}}_+$ by $\overline{\mathbf{F}}_-$.

Proof. Taking u_n as the test function in the variational formulation of the first eigenfunction in (64), we have

$$(\mu_1 - \lambda_n) \int_{\partial\Omega} u_n \varphi_{1,n} = \int_{\partial\Omega} [g(\cdot, u_n) - g_u(\cdot, u_n) u_n] \varphi_{1,n},$$

with $\varphi_{1,n} = \varphi_1(\lambda_n, u_n)$. Now,

$$\frac{\displaystyle\int_{\partial\Omega} [g(\cdot, u_n) - g_u(\cdot, u_n) u_n] \varphi_{1,n}}{\|u_n\|^{\rho}_{L^\infty(\partial\Omega)}} = \int_{\partial\Omega} \frac{g(\cdot, u_n) - g_u(\cdot, u_n) u_n}{|u_n|^\rho} \left(\frac{|u_n|}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^\rho \varphi_{1,n}.$$

Let us observe that from the hypothesis (H3), using that $\Phi_1 > 0$ for all x on $\partial\Omega$ and (6), we obtain

$$\int_{\partial\Omega} \left| \frac{g(\cdot, u_n) - g_u(\cdot, u_n) u_n}{|u_n|^\rho} \left[\left(\frac{|u_n|}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^\rho - \Phi_1^\rho \right] \varphi_{1,n} \right| \leq$$

$$\leq C \left\| \left(\frac{|u_n|}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^\rho - \Phi_1^\rho \right\|_{L^\infty(\partial\Omega)} \to 0, \text{ as } \lambda_n \to \sigma_1.$$

From (68) and hypothesis (H3), we get

$$\int_{\partial\Omega} \left| \frac{g(\cdot, u_n) - g_u(\cdot, u_n) u_n}{|u_n|^\rho} \right| \Phi_1^\rho \, |\varphi_{1,n} - \Phi_1| \leq C \|\varphi_{1,n} - \Phi_1\|_{L^\infty(\partial\Omega)} \to 0, \text{ as } \lambda_n \to \sigma_1.$$

Moreover, using Fatou's Lemma and the definition of $\underline{\mathbf{F}}_+$, we can write

$$\begin{aligned}
\liminf_{n\to\infty} \quad & \int_{\partial\Omega} \quad \frac{g(\cdot, u_n) - g_u(\cdot, u_n) u_n}{|u_n|^\rho} \left(\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^\rho \varphi_{1,n} \\
\geq \quad & \lim_{n\to\infty} \int_{\partial\Omega} \frac{g(\cdot, u_n) - g_u(\cdot, u_n) u_n}{|u_n|^\rho} \left[\left(\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^\rho - \Phi_1^\rho \right] \varphi_{1,n} \\
& + \lim_{n\to\infty} \int_{\partial\Omega} \frac{g(\cdot, u_n) - g_u(\cdot, u_n) u_n}{|u_n|^\rho} \Phi_1^\rho \left(\varphi_{1,n} - \Phi_1 \right) \\
& + \liminf_{n\to\infty} \int_{\partial\Omega} \frac{g(\cdot, u_n) - g_u(\cdot, u_n) u_n}{|u_n|^\rho} \Phi_1^{1+\rho} \\
\geq \quad & \int_{\partial\Omega} \liminf_{n\to\infty} \frac{g(\cdot, u_n) - g_u(\cdot, u_n) u_n}{|u_n|^\rho} \Phi_1^{1+\rho} \geq \underline{\mathbf{F}}_+ .
\end{aligned}$$

The other inequality is obtained in a similar way. This concludes the proof of the lemma. ☑

With this result, we can proceed now to analyze the stability properties of the solutions of (2) bifurcating from infinity. The first result provides sufficient conditions for the stability of positive solutions of (2) bifurcating from infinity. It also states that, under those hypotheses, the stable branch is subcritical and unique in a neighborhood of σ_1. In other words, as $\lambda \to \sigma_1$ the branch of unbounded positive solutions of (2) is composed of stable subcritical solutions and u_λ is unique for each λ.

Theorem 4.5 (stability for subcritical equilibria bifurcating from infinity)**.** *Assume the nonlinearity g satisfies hypotheses* (H1)*,* (H2) *and* (H3)*.*

Assume also the following condition holds

$$\underline{\mathbf{F}}_+ \; > 0. \tag{71}$$

Then, for λ in a neighborhood of σ_1, the following assertions hold.

(i) *The bifurcation from infinity of positive solutions of* (2) *at $\lambda = \sigma_1$ is subcritical,*

(ii) *The positive solution of* (2) *in the branch bifurcating from infinity is unique for each fixed $\lambda \approx \sigma_1$. That is, there exists a small $\delta > 0$ and a large number $M > 0$ such that for each $\sigma_1 - \delta < \lambda < \sigma_1$, there is a unique positive solution of* (2) *u_λ with $\|u_\lambda\|_{L^\infty(\partial\Omega)} \geq M$.*

Even more, this solution is asymptotically stable and its basin of attraction includes all initial conditions which are large enough, i.e., satisfying $\|u_0\|_{L^\infty(\partial\Omega)} \geq M$, with M large enough and uniform for all $\sigma_1 - \delta < \lambda < \sigma_1$.

An analogous result holds for negative solutions under the assumption $\underline{\mathbf{F}}_- > 0$.

Proof. We first prove that $\underline{\mathbf{F}}_+ > 0$ implies $\underline{\mathbf{G}}^\rho_+ > 0$, which implies that the bifurcation is subcritical (see Theorem 2.14). Let us consider $\varepsilon > 0$ a small number. Now, for $x \in \partial\Omega$ fixed, we have

$$\frac{\partial}{\partial s}\left[\frac{g(x,s)}{s}\right] = -\frac{g(x,s) - s g_u(x,s)}{s^2},$$

and if we define

$$\underline{F}_+(x) := \liminf_{s \to +\infty} \frac{s g(x,s) - s^2 g_u(x,s)}{|s|^{1+\rho}},$$

we will have that, as $\lambda \to \sigma_1$, for sufficiently large $s > 0$ and $x \in \partial\Omega$

$$\frac{\partial}{\partial s}\left[\frac{g(x,s)}{s}\right] \leq -s^{\rho-2}[\underline{F}_+(x) - \varepsilon].$$

Integrating now from s to s_1 we deduce

$$\frac{g(x,s_1)}{s_1} - \frac{g(x,s)}{s} \leq \frac{\underline{F}_+(x) - \varepsilon}{1-\rho}\left(s_1^{\rho-1} - s^{\rho-1}\right).$$

Letting $s_1 \to \infty$ for fixed $x \in \partial\Omega$, we have $\frac{g(x,s_1)}{s_1} \to 0$, and then

$$\frac{g(x,s)}{s^\rho} \geq \frac{\underline{F}_+(x) - \varepsilon}{1-\rho}.$$

Passing to the limit as $\lambda \to \sigma_1$ and $s \to \infty$, we get

$$\liminf_{s\to+\infty} \frac{sg(x,s)}{|s|^{1+\rho}} \geq \frac{\underline{F}_+(x) - \varepsilon}{1-\rho}, \qquad \forall x \in \partial\Omega \tag{72}$$

Moreover, since (72) is valid for all $\varepsilon > 0$ arbitrarily small, we will have

$$\liminf_{s\to+\infty} \frac{sg(x,s)}{|s|^{1+\rho}} \geq \frac{\underline{F}_+(x)}{1-\rho}, \qquad \forall x \in \partial\Omega.$$

Multiplying by $\Phi_1^{1+\rho}$ and integrating on $\partial\Omega$ we obtain, from (25),

$$\underline{\mathbf{G}}_+^\rho = \int_{\partial\Omega} \liminf_{\substack{\lambda\to\sigma_1\\ s\to+\infty}} \frac{sg(x,s)}{|s|^{1+\rho}}\ \Phi_1^{1+\rho} \geq \frac{\underline{\mathbf{F}}_+}{1-\rho} > 0.$$

Let us now prove that any positive solution of (2) bifurcating from infinity is stable. For this we follow the argument sketched at the beginning of this Section. Let us denote by $u_\lambda > 0$ a solution of (2) bifurcating from infinity. The eigenvalue problem associated to the linearization around u_λ, is given by (63) .

Hence, we will show that the first eigenvalue is positive for λ close enough to σ_1. To do that we note that with the notations in (42) we have that the first eigenvalue of (63) can be written as $\Lambda_1 = \Lambda_1(-\lambda - g_u(x, u_\lambda))$. Then, we consider first eigenvalue μ_1 of the auxiliary Steklov linearized eigenvalue problem (64). Then, in (64), the notations in (42) imply that the first *interior* eigenvalue satisfies $\Lambda_1(-\mu_1 - g_u(x, u_\lambda)) = 0$. As we show below that $\mu_1 > \lambda$, we get then $\Lambda_1 = \Lambda_1(-\lambda - g_u(x, u_\lambda)) > 0$ and obtain the stability. Hence, to conclude the proof note that using Lemma 4.4 and the hypothesis (71) we have

$$\liminf_{\lambda\to\sigma_1} \frac{\mu_1 - \lambda}{\|u_\lambda\|^{\rho-1}_{L^\infty(\partial\Omega)}} \;\geq\; \frac{\underline{\mathbf{F}}_+}{\int_{\partial\Omega} {\Phi_1}^2} \;>\; 0,$$

and therefore $\mu_1 > \lambda$ for λ close enough to σ_1.

We will now prove uniqueness of large solutions of (2) for fixed λ close to σ_1. From the previous results there exists a $\delta > 0$ small enough, and $M > 0$ large enough, such that for $\lambda \in (\sigma_1 - \delta, \sigma_1)$, there exists at least one solution of (2) with $u_\lambda > 0$ and $\|u_\lambda\|_{L^\infty(\partial\Omega)} \geq M$, and also any such solution is asymptotically stable. Moreover, from (6) and (7), and maybe choosing a smaller $\delta > 0$, we have that any positive solution of (2) u bifurcating from infinity actually satisfies $u(x) > M$ for all $x \in \bar{\Omega}$. Let us denote by $\mathcal{E}_\lambda$ the set of solutions of (2) satisfying $u(x) > M$ for all $x \in \bar{\Omega}$. Our objective is to show that $\mathcal{E}_\lambda$ is a singleton.

Since all solutions in $\mathcal{E}_\lambda$ are asymptotically stable, we will have only a finite number of them. Moreover, applying [7, Proposition 7.1], we will have that for fixed $\lambda \in (\sigma_1 - \delta, \sigma_1)$ there exists a maximal solution in $\mathcal{E}_\lambda$, that is, there exists $u_\lambda \in \mathcal{E}_\lambda$ such that for any other $v \in \mathcal{E}_\lambda$ we have $v \leq u_\lambda$.

Let us assume that there exists $v_0 \in \mathcal{E}_\lambda$ with $v_0 \neq u_\lambda$. By the strong maximum principle, we will have that $v_0(x) < u_\lambda(x)$ for all $x \in \bar{\Omega}$. Moreover, if we define the set $[v_0, u_\lambda] = \{\varphi \in C(\bar{\Omega}), v_0(x) \leq \varphi(x) \leq u_\lambda(x)\}$ we will have that this set is positively invariant under the flow defined by (1), $T_\lambda(t)$. That is, if $T_\lambda(t)\varphi$ denotes the solution of (1) with initial condition φ and if $\varphi \in [v_0, u_\lambda]$, then $T_\lambda(t)\varphi \in [v_0, u_\lambda]$ for all $t > 0$.

Since $T_\lambda(t)$ is a gradient system (see Section 4.6), then $T_\lambda(t)\varphi$ must converge to one of the equilibriums in the interval $[v_0, u_\lambda]$ which we denote $\{v_0, v_1, \ldots, v_{k+1} = u_\lambda\}$.

Let us consider now the convex linear combination of the functions v_0 and u_λ, that is, $\varphi_\eta = (1-\eta)v_0 + \eta v_{k+1} \in [v_0, u_\lambda]$ for $\eta \in [0,1]$. Define the function $h : [0,1] \to \{0, 1, \ldots, k+1\}$ as follows: $h(\eta) = j$ if $T_\lambda(t)\varphi_\eta \to v_j$ as $t \to +\infty$. Observe that this function is well defined and that we have $h(0) = 0$ and $h(1) = k+1$. Moreover since all equilibria are asymptotically stable and using the continuous dependence of the solutions of (1) with respect to initial conditions in finite intervals of time, we can easily show that h is continuous. Hence, it is a constant function, which is impossible since $h(0) = 0$ and $h(1) = k+1$. Therefore, there cannot exist a function v_0 in $\mathcal{E}_\lambda$ different from u_λ.

The global asymptotic stability (with respect to large solutions of (1)) of the unique positive large equilibrium of (2) follows as in the proof of Proposition 7.1 in [7]. ☑

We state now a result on the instability of solutions for the case of a supercritical bifurcation. Now this result provides sufficient conditions for the instability of positive solutions of (2) bifurcating from infinity. It also states that, under those hypotheses, the unstable branch is supercritical and unique in a neighborhood of σ_1. In other words, as $\lambda \to \sigma_1$ the unbounded branch of positive solutions of (2) is composed of unstable supercritical solutions and u_λ is unique for each λ.

Theorem 4.6 (Instability for supercritical equilibria bifurcating from infinity). *Assume the nonlinearity g satisfies hypotheses* (H1), (H2) *and* (H3). *Assume also that the following condition holds*

$$\overline{\mathbf{F}}_+ \ < 0. \tag{73}$$

Then, for λ in a neighborhood of σ_1 the following assertions hold.

(i) *The bifurcation from infinity of positive solutions of* (2) *at $\lambda = \sigma_1$ is supercritical.*

(ii) *The positive equilibrium solution of* (2) *contained in the branch bifurcating from infinity is unique for each λ close enough to σ_1 and it is unstable.*

An analogous result holds for negative solutions of (2) *under the assumption $\overline{\mathbf{F}}_- \ < 0$.*

Proof. To prove that the bifurcation is supercritical we proceed as in the proof of Theorem 4.5. We therefore skip the details here.

To prove the instability, we proceed as in the proof of Theorem 4.5, but now, from Lemma 4.4 we have

$$\limsup_{\lambda\to\sigma_1} \frac{\mu_1 - \lambda}{\|u_\lambda\|^{\rho-1}_{L^\infty(\partial\Omega)}} \le \frac{\overline{\mathbf{F}}_+}{\int_{\partial\Omega} \Phi_1{}^2} < 0,$$

and therefore $\mu_1 < \lambda$ for λ close enough to σ_1, and the equilibrium is unstable.

Now we prove the uniqueness of the solution in the branch. Assume on the contrary that for some sequence $\lambda_n \to \sigma_1$, with $\lambda_n > \sigma_1$, there exist two different supercritical unstable positive solutions of (2), u_n and v_n, satisfying (6).

Note then that u_n and v_n can not be ordered, since otherwise, there would be a stable large solution in between. This would contradict the instability shown above.

Let us define $w_n = u_n - v_n$, w_n which changes sign in Ω. By substracting the equations satisfied by u_n and v_n and taking Φ_1 as a test function, we get

$$(\sigma_1 - \lambda_n) \int_{\partial\Omega} w_n \Phi_1 = \int_{\partial\Omega} \left[g(\lambda_n, \cdot, u_n) - g(\lambda_n, \cdot, v_n) \right] \Phi_1. \tag{74}$$

Let us write

$$g(\lambda_n, x, u_n) - g(\lambda_n, x, v_n) = w_n \int_0^1 g_u(\lambda_n, x, \tau u_n + (1-\tau)v_n)\ d\tau,$$

and set $b_n(x) := \int_0^1 g_u(\lambda_n, x, \tau u_n + (1-\tau)v_n)\ d\tau$. Using (7) and (H2)-(H3), we can assert that

$$b_n \to 0 \text{ in } L^r(\partial\Omega), \text{ with } r > N-1. \tag{75}$$

Set now $z_n = \frac{w_n}{\|w_n\|_{L^\infty(\partial\Omega)}}$. Then, z_n satisfies the following problem

$$\begin{cases} -\Delta z_n + z_n &= 0, & \text{in } \Omega, \\ \dfrac{\partial z_n}{\partial n} &= \lambda_n z_n + b_n(x) z_n, & \text{on } \partial\Omega, \end{cases}$$

with $\|z_n\|_{L^\infty(\partial\Omega)} = 1$.

From here, taking into account that $b_n \in L^r(\partial\Omega)$ for $r > N-1$ (see (75)), and using regularity results for the linear problem (see for instance [7, lemma 2.1]), we then get $\|z_n\|_{C^\nu(\overline{\Omega})} \leq C$ for some $\nu \in (0,1)$. By the compact imbedding $C^\nu(\overline{\Omega}) \hookrightarrow C^\beta(\overline{\Omega})$ for $0 < \beta < \nu$, and taking subsequences if necessary, we can assume that z_n converges to z in $C^\beta(\overline{\Omega})$. Hence $\|z\|_{L^\infty(\partial\Omega)} = 1$. Moreover, using (75), z is an eigenfunction of the Steklov eigenvalue problem (3), associated to the first eigenvalue σ_1 (see Proposition 3.2 in Section 3). Since this is simple, we deduce either $z > 0$ or $z < 0$, and in any case either $z_n > 0$ or $z_n < 0$, or equivalently either $w_n > 0$ or $w_n < 0$, which contradicts the fact that w_n changes sign. Therefore, for λ sufficiently close to σ_1 the solution of (2) bifurcating from infinity is unique. ☑

5. Turning points and the resonant problem

Another interesting question is that of the resonant problem, that is when $\lambda = \sigma_1$. For this case, we obtain in Theorem 5.1, some Landesman–Lazer type conditions guaranteeing that the resonant problem has solution; cf. [26]. In the language of bifurcation, these type of conditions can be stated as: *if all the unbounded branches are either subcritical or supercritical, then the resonant problem has at least one solution.*

In Subsection 5.8 we apply the conditions from the sub-super critical bifurcation (see Subsection 2.3), to obtain Landesman-Lazer type conditions for the resonant problem.

In Subsection 5.9, precisely when Landesman-Lazer type conditions do not hold, we will state sufficient conditions for the existence of an unbounded sequence of infinitely many resonant solutions and infinitely many turning points.

5.8. The resonant case I: at least a resonant solution

We are concerned now with the resonant problem, that is,

$$\begin{cases} -\Delta u + u &= 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} &= \sigma u + g(x,u), & \text{on } \partial\Omega, \end{cases} \tag{76}$$

where σ is an Steklov eigenvalue of (3). We are interested in giving conditions guaranteeing the existence of solutions in this case. As a matter of fact, we will see that if all possible bifurcations of the problem

$$\begin{cases} -\Delta u + u &= 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} &= \lambda u + g(x,u), & \text{on } \partial\Omega, \end{cases} \tag{77}$$

with $\lambda \in \mathbb{R}$, $\lambda \approx \sigma$ are either subcritical or supercritical, then the resonant problem necessarily has at least a solution.

Theorem 5.1. *Assume that condition* (31) *holds, that is, every possible bifurcation from infinity at $\lambda = \sigma$ of problem* (77) *is subcritical, or condition* (32) *holds, that is, every possible bifurcation from infinity at $\lambda = \sigma$ of problem* (77) *is supercritical. Then, the resonant problem* (76) *has at least one solution.*

Remark 5.2. Conditions (31) and (32) are known as Landesman-Lazer type conditions.

Proof. Observe first that from Theorem 2.7, for $\epsilon > 0$ small enough, we have that problem (77) has at least one solution for all $\lambda \in (\sigma - \epsilon, \sigma + \epsilon) \setminus \{\sigma\}$. If, for instance, we assume that all possible bifurcations occurring at $\lambda = \sigma$ are subcritical, then necessarily there exists a constant M such that for any $\lambda \in (\sigma, \sigma + \epsilon)$ all possible solutions of (77) satisfy $\|u\|_{L^\infty(\partial\Omega)} \leq M$. This allows us to take a sequence of $\lambda_n \to \sigma$ and solutions u_n of (77) with $\|u_n\|_{L^\infty(\partial\Omega)} \leq M$. Using the compactness given by elliptic regularity results applied to (77), and passing to the limit, we obtain a solution of (76). ☑

5.9. The resonant case I: Infinitely many resonant solutions

In this Subsection we consider nonlinearities for which

$$\underline{\mathbf{G}}_+ < 0 < \overline{\mathbf{G}}_+,$$

a condition that somehow reflects some oscillatory character of the nonlinear term at infinity, which we hope to translate into an oscillatory behavior of the bifurcating branches. Observe that in this situation, both the criteria for sub/super criticality and the Landesman–Lazer type conditions do not hold.

In such a situation our goal is threefold: first we give easy–to–check conditions on the nonlinear term, guaranteeing that in $\mathcal{D}^+$ there are large subcritical and supercritical solutions.

Second, the connectedness of $\mathcal{D}^+$ suggests that we would be able to find an unbounded sequence of *turning points* (see Definition 1.1).

Note that, generically, in a neighborhood of a turning point there are, at least, two solutions for the same value of the parameter *at one side*, either $\lambda < \lambda^*$, or either for $\lambda > \lambda^*$. Therefore, turning points are related with multiplicity of solutions.

Third, the connectedness of $\mathcal{D}^+$ suggests that we would be able to find an unbounded sequence of resonant solutions.

Related results for the case of an interior reaction term were established in [10], [14] and [19].

In [10] the author works in a one dimensional problem with an interior reaction term of the type $g(s) + a\sin(x) + h(x)$, where g is an s-periodic function of zero mean and h is orthogonal to the first corresponding eigenfunction. He proves that as the problem approaches resonance, the number of solutions increases to infinity.

In [14] the authors work in domains $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ satisfying some geometric condition, and with a nonlinearity of the type $g(s) + h(x)$, where g and h are as before. This geometric condition is accomplished in balls when $N = 2$ and also, when $N \geq 2$, in annular domains $\{x : a < |x| < b\}$ with a large and $b - a$ small. They proved that the resonant problem has infinite solutions.

In [19] the authors work in the unit ball $B \subset \mathbb{R}^N$ with $N \geq 1$. They proved that the resonant problem has infinite solutions for $1 \leq N \leq 5$ and at most finitely many solutions for $N \geq 6$.

This Section is organized as follows. In Sec. 5.10 we make precise the hypotheses on the nonlinearity and present in more detail the techniques we use and the main results. In Sec. 5.11 we prove the main result of the section, Theorem 5.6, which gives the existence of unbounded sequences of turning points and resonant solutions. In Sec. 5.12 we illustrate our results with two examples, where we consider nonlinear terms of the type

$$g(x,s) := s^\alpha \left[\sin\left(\left| \frac{s}{\Phi_1(x)} \right|^\beta \right) + C \right], \tag{78}$$

with $\alpha < 1$, $\beta > 0$, $\alpha + \beta < 1$ and $|C| < 1$.

5.10. Introduction on oscillatory branches

With respect to the nonlinearity g in (2), we assume the hypotheses (H1) and (H2).

Note that solutions of (2) are determined and estimated in terms of their boundary values. Therefore, we can look at (2) as a problem posed in a space of functions defined on $\partial\Omega$.

Now we describe the technique we follow to prove the main result, Theorem 5.6. Note that this result gives easy–to–check conditions on the nonlinear term, guaranteeing that in $\mathcal{D}^+$ there are large subcritical and supercritical solutions. We start out of (5), from where we know that for $(\lambda, u) \in \mathcal{D}^+$, with $\lambda \to \sigma_1$, we have

$$u = s\Phi_1 + w,$$

where

$$\int_{\partial\Omega} w\Phi_1 = 0 \qquad \text{and} \quad w = o(s) \text{ as } s \to \infty.$$

With this, we are able to prove that if

$$|g(x,s)| = O\left(|s|^{\alpha}\right) \qquad \text{as } s \to \infty,$$

then

$$w = O(|s|^{\alpha}) \quad \text{and} \quad |\sigma_1 - \lambda| = O(|s|^{\alpha-1})$$

as $s \to \infty$ (see Proposition 5.4).

Now, consider a sequence $(\lambda_n, u_n) \in \mathcal{D}^+$ with

$$\lambda_n \to \sigma_1 \quad \text{and} \quad \|u_n\|_{L^\infty(\partial\Omega)} \to \infty.$$

Using the results in Lemma 2.13, to determine whether a sequence of solutions lies at one side or another of σ_1 one must check the sign of

$$\liminf_{n\to\infty} \int_{\partial\Omega} \frac{u_n g(\cdot, u_n)}{|u_n|^{1+\alpha}} \Phi_1^{1+\alpha}, \tag{79}$$

and of

$$\limsup_{n\to\infty} \int_{\partial\Omega} \frac{u_n g(\cdot, u_n)}{|u_n|^{1+\alpha}} \Phi_1^{1+\alpha} \tag{80}$$

(see also [8, Lemma 3.1]). But this requires a knowledge of the solutions themselves.

Using the previous results, we write

$$u_n = s_n \Phi_1 + w_n,$$

where

$$\int_{\partial\Omega} w_n \Phi_1 = 0 \quad \text{and } w_n = O(|s_n|^{\alpha}) \text{ as } n \to \infty,$$

and we intend to unveil the signs in (79) by just looking at the signs of

$$\liminf_{n\to\infty} \int_{\partial\Omega} \frac{s_n g(\cdot, s_n\Phi_1)}{|s_n|^{1+\alpha}} \Phi_1, \tag{81}$$

and of

$$\limsup_{s_n\to\infty} \int_{\partial\Omega} \frac{s_n g(\sigma_1, \cdot, s_n\Phi_1)}{|s_n|^{1+\alpha}} \Phi_1. \tag{82}$$

This is achieved in Lemma 6.4.

With these tools, in Theorem 5.6 we take two sequences $\{s_n\}$ and $\{s'_n\}$ satisfying

$$0 < \lim_{n\to+\infty} \int_{\partial\Omega} s_n \frac{g(\cdot, s_n\Phi_1)}{|s_n|^{1+\alpha}} \Phi_1 \ < \infty,$$

$$-\infty < \lim_{n\to+\infty} \int_{\partial\Omega} \frac{s'_n g(\cdot, s'_n\Phi_1)}{|s'_n|^{1+\alpha}} \Phi_1 \ < 0,$$

and from here we obtain the existence of unbounded sequences of sub and supercritical solutions of (2) in $\mathcal{D}^+$.

Finally exploiting the connectedness of $\mathcal{D}^+$, we obtain the existence of unbounded sequences of turning points and of resonant solutions.

5.11. Infinitely many turning points and infinitely many resonant solutions

In this section we give sufficient conditions for the existence of a branch of solutions bifurcating from infinity which is neither subcritical nor supercritical. From this, we conclude the existence of infinitely many *turning points* (see Definition 1.1), and an infinite number of solutions for the resonant problem, i.e., for $\lambda = \sigma_1$. This is achieved in Theorem 5.6

For this we first consider a family of linear Steklov problems with a variable nonhomogeneous term at the boundary h depending on the parameter λ

$$\left\{ \begin{array}{rcll} -\Delta u + u & = & 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} & = & \lambda u + h(x), & \text{on } \partial\Omega, \end{array} \right. \tag{83}$$

where $h \in L^r(\partial\Omega)$, $r > N-1$ and $\lambda \in (-\infty, \sigma_2)$.

We use now the decomposition

$$L^r(\partial\Omega) = \operatorname{span}[\Phi_1] \oplus \operatorname{span}[\Phi_1]^\perp, \tag{84}$$

where

$$\operatorname{span}[\Phi_1]^\perp := \left\{ u \in L^r(\partial\Omega) : \int_{\partial\Omega} u\,\Phi_1 = 0 \right\},$$

and then for $h \in L^r(\partial\Omega)$, with $r > N-1$, there exists a unique decomposition

$$h = a_1 \Phi_1 + h_1, \tag{85}$$

where

$$a_1 = \frac{\int_{\partial\Omega} h\,\Phi_1}{\int_{\partial\Omega} {\Phi_1}^2} \quad \text{and} \quad \int_{\partial\Omega} h_1 \Phi_1 = 0.$$

The Fredholm Alternative states that the linear problem (83) has a unique solution if $\lambda \neq \sigma_1$, and does not have solution if

$$\lambda = \sigma_1$$

and

$$a_1 \neq 0.$$

Hence, for $\lambda \neq \sigma_1$ the solution $u = u(\lambda)$ of (83) has a unique decomposition

$$u = \frac{a_1}{\sigma_1 - \lambda}\,\Phi_1 + w, \tag{86}$$

where

$$\int_{\partial\Omega} w\Phi_1 = 0, \quad \text{and } w = w(\lambda) \in \operatorname{span}[\Phi_1]^\perp \tag{87}$$

solves the problem

$$\left\{ \begin{array}{rcll} -\Delta w + w & = & 0, & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} & = & \lambda w + h_1(x), & \text{on } \partial\Omega. \end{array} \right. \tag{88}$$

Note that in (86) the solution $w(\lambda) \in \operatorname{span}[\Phi_1]^\perp$ is also well defined for $\lambda = \sigma_1$.

The next result states that $w = w(\lambda)$ is uniformly bounded if h is so.

Lemma 5.3. *For each compact set $K \subset (-\infty, \sigma_2) \subset \mathbb{R}$ there exists a constant $C = C(K)$, independent of λ, such that for any $\lambda \in K$*

$$\|w(\lambda)\|_{L^\infty(\partial\Omega)} \le C\|h\|_{L^r(\partial\Omega)},$$

where $w \in span[\Phi_1]^\perp$ is the solution of (86) and $h_1 \in span[\Phi_1]^\perp$ is defined in (85).

Proof. Note again that by the Fredholm Alternative the solution of (86), $w = w(\lambda) \in \text{span}[\Phi_1]^\perp$, is well defined for any $\lambda \in K$.

First, we prove that $w(\lambda)$ is uniformly bounded for any λ in a neighborhood of σ_1. Assume this is not the case. Then, there is a sequence $\lambda_n \to \sigma_1$ with $\|w(\lambda_n)\|_{L^\infty(\partial\Omega)} \to \infty$. From [7, Corollary 3.2], we have

$$\frac{w(\lambda_n)}{\|w(\lambda_n)\|_{L^\infty(\partial\Omega)}} \to \Phi_1 \quad \text{uniformly in } \overline{\Omega},$$

contradicting the fact that $w(\lambda) \in \text{span}[\Phi_1]^\perp$. Therefore, there exists some $\delta > 0$ such that

$$\|w(\lambda)\|_{L^\infty} < C \quad \text{independent of } \lambda$$

for any $|\lambda - \sigma_1| < \delta$.

Second,

$$\|w(\lambda)\|_{L^\infty} < \infty \text{ for any } \lambda \in K \setminus (\sigma_1 - \delta, \sigma_1 + \delta),$$

since the linear operator is invertible (see Theorem 2.7 in [7]).

Now we define the family of operators

$$T(\lambda) : L^r(\partial\Omega) \to L^\infty(\partial\Omega)$$

by

$$T(\lambda)h := w(\lambda).$$

Then, $T(\lambda)$ is continuous for every $\lambda \in K$, and

$$\sup_{\lambda \in K} \|T(\lambda)h\|_{L^\infty(\partial\Omega)} < \infty.$$

Therefore, applying the uniform boundedness principle, there exists a constant $C = C(K)$ such that

$$\|w(\lambda)\|_{L^\infty(\partial\Omega)} \le C(K)\|h\|_{L^r(\partial\Omega)}$$

for any $\lambda \in K$, and we get the result. ☑

Now we turn into the nonlinear problem (2). Recall that for λ close to σ_1 we have (5), that is, as $\lambda \to \sigma_1$ the unbounded solutions satisfy

$$u = s\Phi_1 + w, \quad \text{where } w \in \text{span}[\Phi_1]^\perp, \tag{89}$$

$$w = o(s), \quad \text{as } s \to \infty. \tag{90}$$

For later use, we define

$$P(u) = \frac{\int_{\partial\Omega} u\, \Phi_1}{\int_{\partial\Omega} \Phi_1^2}. \tag{91}$$

Then, we give conditions on the nonlinear term g in (2), guaranteeing that in (89) the order of w in (89) is $w = O(|s|^\alpha)$ as $s \to \infty$. Note that we restrict ourselves below to the unbounded branch of positive solutions. A completely analogous result holds for the unbounded branch of negative solutions.

Proposition 5.4. *Assume g satisfies hypotheses* (H1) *and* (H2) *for $\sigma = \sigma_1$.*

Then, there exists an open set $\mathcal{O} \subset \mathbb{R} \times C(\bar{\Omega})$ of the form

$$\mathcal{O} = \{(\lambda, u) : |\lambda - \sigma_1| < \delta_0,\ u(x) > M_0\}$$

for some small δ_0 and large M_0, such that $\mathcal{D}^+$, the unbounded branch of positive solutions of (2)*, satisfies*

(i) *There exists a constant C_1 independent of λ such that, if $(\lambda, u) \in \mathcal{D}^+ \cap \mathcal{O}$ and $(\lambda, u) \neq (\sigma_1, \infty)$, then $u = s\Phi_1 + w$, where $s > 0$, $w \in span[\Phi_1]^\perp$ and*

$$\|w\|_{L^\infty(\partial\Omega)} \leq C_1 \|G_1\|_{L^r(\partial\Omega)}\, |s|^\alpha$$

as $|s| \to \infty$.

(ii) *There exists some constant $S_0 > 0$ such that for all $s \geq S_0$ there exists a solution $(\lambda, u) \in \mathcal{D}^+ \cap \mathcal{O}$ satisfying*

$$u = s\Phi_1 + w, \quad \textit{with } w \in span[\Phi_1]^\perp.$$

(iii) *Moreover, there exists a constant C_2 independent of λ such that, for any solution of the type $(\lambda, u) \in \mathcal{D}^+ \cap \mathcal{O}$, $u = s\Phi_1 + w$, with $w \in span[\Phi_1]^\perp$, the following holds*

$$|\sigma_1 - \lambda| \leq C_2 |s|^{\alpha - 1}, \qquad \textit{as } |s| \to \infty,$$

with

$$C_2 = \frac{2\|G_1\|_{L^1(\partial\Omega)}}{\int_{\partial\Omega} \Phi_1^2}.$$

Proof. Note that (H1), Lemma 5.3 and the fact that, from (89),

$$\Phi_1 + w/s \to \Phi_1 \quad \text{as } s \to \infty$$

in $L^\infty(\partial\Omega)$, imply that in fact that

$$\|w\|_{L^\infty(\partial\Omega)} \leq C|s|^\alpha \text{ as } s \to \infty.$$

This proves part (i).

To prove part (ii) note that $\mathcal{D}^+ \cap \mathcal{O}$, although not necessarily connected, it has an unbounded connected component. Hence, using the decomposition (89), we have $u =$

$s\Phi_1 + w$ with $w \in \mathrm{span}[\Phi_1]^\perp$. Since the projection (91) is continuous, the set $s \in \mathbb{R}$ such that there exists a solution of (2) with $u = s\Phi_1 + w$ with $w \in \mathrm{span}[\Phi_1]^\perp$ contains an unbounded connected set in $\mathbb{R}$.

To prove part (iii), we observe that if (λ, u) is a solution of (2), $u = s\Phi_1 + w$, with $w \in \mathrm{span}[\Phi_1]^\perp$, multiplying the equation by the first Steklov eigenfunction $\Phi_1 > 0$ and integrating by parts we obtain

$$(\sigma_1 - \lambda)s \int_{\partial\Omega} \Phi_1^2 = \int_{\partial\Omega} g(x, s\Phi_1 + w)\Phi_1. \tag{92}$$

Taking into account that

$$\frac{|g(x, s\Phi_1 + w)|}{|s|} = \frac{|g(x, s\Phi_1 + w)|}{|s\Phi_1 + w|} \left|\Phi_1 + \frac{w}{s}\right|, \tag{93}$$

and that

$$\frac{|g(x, s\Phi_1 + w)|}{|s\Phi_1 + w|} \to 0 \quad \text{as } s \to \infty, \tag{94}$$

we get

$$\lambda \to \sigma_1 \quad \text{as } s \to \infty. \tag{95}$$

Moreover, from (H1), we obtain that

$$\begin{aligned} |g(x, s\Phi_1 + w)| &= |s|^\alpha \frac{|g(x, s\Phi_1 + w)|}{|s\Phi_1 + w|^\alpha} \left|\Phi_1 + \frac{w}{s}\right|^\alpha \\ &\leq |s|^\alpha G_1(x) \left|\Phi_1 + \frac{w}{s}\right|^\alpha, \end{aligned} \tag{96}$$

and therefore

$$\begin{aligned} |\sigma_1 - \lambda| &\leq \frac{|s|^{\alpha-1}}{\int_{\partial\Omega} \Phi_1^2} \int_{\partial\Omega} G_1(x) \left|\Phi_1 + \frac{w}{s}\right|^\alpha \Phi_1 \\ &\leq C\|G_1\|_{L^r(\partial\Omega)} |s|^{\alpha-1}, \end{aligned}$$

which ends the proof. ☑

After this, in order to prove the main result, Theorem 5.6 below, we need to guarantee that the signs in (79) can be determined by the signs in (5.10), that is,

$$\liminf_{(\lambda,s)\to(\sigma_1,+\infty)} \int_{\partial\Omega} \frac{sg(\cdot, s\Phi_1)}{|s|^{1+\alpha}} \Phi_1 \quad < 0 < \quad \limsup_{(\lambda,s)\to(\sigma_1,+\infty)} \int_{\partial\Omega} \frac{sg(\cdot, s\Phi_1)}{|s|^{1+\alpha}} \Phi_1. \tag{97}$$

In order to guarantee that (97) is enough to conclude the existence of sub and supercritical solution in the unbounded branch, we will use the following result.

Lemma 5.5. *Given $h(x,s)$, differentiable with respect to the last variable, assume that for some $\alpha < 1$ there exists a function H_1 such that for all $s \approx +\infty$ and $x \in \partial\Omega$ we have*

$$\left|\frac{h(x,s)}{|s|^\alpha}\right| \leq H_1(x), \qquad H_1 \in L^1(\partial\Omega). \tag{98}$$

Assume also its partial derivative $\frac{\partial h}{\partial s}(\cdot,\cdot) \in C(\partial\Omega \times \mathbb{R})$*, and*

$$\sup_{|s|\geq M} \left\| \frac{\partial h}{\partial s}(\cdot, s) \right\|_{L^\infty(\partial\Omega)} \to 0 \tag{99}$$

as $M \to +\infty$*.*

Let $\lambda_n \to \sigma_1$*,* $s_n \uparrow \infty$ *and* w_n *in* $L^\infty(\partial\Omega)$*, such that*

$$\|w_n\|_{L^\infty(\partial\Omega)} \leq C|s_n|^\alpha$$

as $n \to \infty$ *for some constant* C*. Then, the following holds*

$$\liminf_{n\to+\infty} \int_{\partial\Omega} \frac{(s_n\Phi_1 + w_n)h(\cdot, s_n\Phi_1 + w_n)}{|s_n\Phi_1 + w_n|^{1+\alpha}}\ \Phi_1^{1+\alpha} \geq \liminf_{n\to+\infty} \int_{\partial\Omega} \frac{s_n h(\cdot, s_n\Phi_1)}{|s_n|^{1+\alpha}}\ \Phi_1,$$

and similarly

$$\limsup_{n\to+\infty} \int_{\partial\Omega} \frac{(s_n\Phi_1 + w_n)h(\cdot, s_n\Phi_1 + w_n)}{|s_n\Phi_1 + w_n|^{1+\alpha}}\ \Phi_1^{1+\alpha} \leq \limsup_{n\to+\infty} \int_{\partial\Omega} \frac{s h(\cdot, s_n\Phi_1)}{|s_n|^{1+\alpha}}\ \Phi_1.$$

Proof. For all $(\lambda, s) \approx (\sigma_1, +\infty)$ and for any $w \in L^\infty(\partial\Omega)$ such that $\frac{1}{2}\Phi_1 > \frac{|w|}{s}$, we have (with a constant C that may change from line to line)

$$\begin{aligned}
\int_{\partial\Omega} |h(\cdot, s\Phi_1 + w) - h(\cdot, s\Phi_1)|\Phi_1 \quad &\leq \quad C\|w\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \left| \int_0^1 \frac{\partial h}{\partial s}(\cdot, s\Phi_1 + \tau w)\, d\tau \right| \\
&\leq \quad C\|w\|_{L^\infty(\partial\Omega)} \sup_{\tau\in[0,1]} \left\| \frac{\partial h}{\partial s}(\cdot, s\Phi_1 + \tau w) \right\|_{L^\infty(\partial\Omega)}.
\end{aligned}$$

Taking into account hypothesis (99), and whenever $\|w\|_{L^\infty(\partial\Omega)} = O(|s|^\alpha)$, we deduce that

$$\int_{\partial\Omega} \frac{|h(\cdot, s\Phi_1 + w) - h(\cdot, s\Phi_1)|}{|s|^\alpha}\ \Phi_1 \leq C \sup_{|s|\geq M} \left\| \frac{\partial h}{\partial s}(\cdot, s) \right\|_{L^\infty(\partial\Omega)} \longrightarrow 0, \tag{100}$$

as $\lambda \to \sigma_1$, $M \to +\infty$.

Consequently, for $\|w_n\|_{L^\infty(\partial\Omega)} = O(|s_n|^\alpha)$

$$\begin{aligned}
&\liminf_{n\to+\infty} \int_{\partial\Omega} \frac{s_n h(\cdot, s_n\Phi_1 + w_n)}{|s_n|^{1+\alpha}}\ \Phi_1 \\
&\quad \geq \lim_{\substack{\lambda\to\sigma_1 \\ s\to+\infty}} \int_{\partial\Omega} \frac{s h(\cdot, s\Phi_1 + w) - s h(\cdot, s\Phi_1)}{|s|^{1+\alpha}}\ \Phi_1 + \liminf_{n\to+\infty} \int_{\partial\Omega} \frac{s_n h(\cdot, s_n\Phi_1)}{|s_n|^{1+\alpha}}\ \Phi_1 \qquad (101) \\
&\quad = \liminf_{n\to+\infty} \int_{\partial\Omega} \frac{s_n h(\cdot, s_n\Phi_1)}{|s_n|^{1+\alpha}}\ \Phi_1,
\end{aligned}$$

where we used (100).

Now note that the left hand side above can be written as

$$\frac{s_n h(\cdot, s_n\Phi_1 + w_n)}{|s_n|^{1+\alpha}}\ \Phi_1 = \frac{(s_n\Phi_1{+}w_n)h(\cdot, s_n\Phi_1 + w_n)}{|s_n\Phi_1{+}w_n|^{1+\alpha}}\left|\Phi_1 + \frac{w_n}{s_n}\right|^{\alpha}\Phi_1.$$

Then, (98) and the fact that $\Phi_1 + w_n/s_n \to \Phi_1$ in $L^\infty(\partial\Omega)$ conclude the proof. ☑

Now we are in a position to prove the main result in this section that, roughly speaking, states that if there are a sequence of subcritical solutions and another of supercritical solutions, since the solution set is connected, there are infinite turning points and infinite resonant solutions. We state the result for the positive branch. The same conclusions can be attained for the connected branch of negative solutions bifurcating from infinity.

Theorem 5.6. *Assume the nonlinearity g satisfies hypotheses* (H1) *and* (H2)*. Assume that the nonlinearity $g_s(x,s)$ is differentiable in s and its partial derivative $g_s(\cdot,\cdot) \in C(\partial\Omega\times\mathbb{R})$. Assume also that*

$$\sup_{|s|\geq M}\left\|\frac{\partial g}{\partial s}(\cdot,s)\right\|_{L^\infty(\partial\Omega)} \to 0 \tag{102}$$

as $M \to +\infty$.

Assume, moreover, that there exist two increasing sequences $\{s_n\}$, $\{s'_n\}$ both convergent to $+\infty$, such that

$$0 < \lim_{n\to+\infty}\int_{\partial\Omega} s_n \frac{g(\cdot, s_n\Phi_1)}{|s_n|^{1+\alpha}}\Phi_1 \ < \infty, \tag{103}$$

and

$$-\infty < \lim_{n\to+\infty}\int_{\partial\Omega} s'_n \frac{g(\cdot, s'_n\Phi_1)}{|s'_n|^{1+\alpha}}\Phi_1 \ < 0 \tag{104}$$

Then, in the connected branch of positive solutions bifurcating from infinity, $\mathcal{D}^+$, the following assertions hold.

(i) For sufficiently large $n \gg 1$, any solution (λ, u) is subcritical if

$$P(u) = \frac{\int_{\partial\Omega} u\Phi_1}{\int_{\partial\Omega}\Phi_1^2} = s_n,$$

and supercritical if $P(u) = s'_n$. Consequently, there exist two sequences of solutions of (2), $\{(\lambda_n, u_n)\}$ and $\{(\lambda'_n, u'_n)\}$ converging to (σ_1, ∞) as $n \to \infty$, one of them subcritical, $\lambda_n < \sigma_1$, and the other supercritical, $\lambda'_n > \sigma_1$.

(ii) There is an unbounded sequence of turning points $\{(\lambda^*_n, u^*_n)\}$ such that

$$\lambda^*_n \to \sigma_1, \quad \|u^*_n\|_{L^\infty(\partial\Omega)} \to \infty, \quad \textit{as } n\to\infty.$$

In fact, we can always choose two subsequences of turning points, one of them subcritical, $\lambda^*_{2n+1} < \sigma_1$, and the other supercritical, $\lambda^*_{2n} > \sigma_1$.

(iii) There is an unbounded sequence of resonant solutions, i.e., there are infinite solutions $\{(\sigma_1, \widetilde{u}_n)\}$ *of* (2) *with* $\|\widetilde{u}_n\|_{L^\infty(\partial\Omega)} \to \infty$.

Proof. From Proposition 5.4, (ii), consider any two sequences of solutions of (2), such that $(\lambda_n, u_n) \to (\sigma_1, \infty)$ and $(\lambda_n', u_n') \to (\sigma_1, \infty)$ in $\mathcal{D}^+$ with

$$P(u_n) = \frac{\int_{\partial\Omega} u_n \Phi_1}{\int_{\partial\Omega} \Phi_1^2} = s_n,$$

and

$$P(u_n') = \frac{\int_{\partial\Omega} u_n' \Phi_1}{\int_{\partial\Omega} \Phi_1^2} = s_n'.$$

Writing $u_n = s_n\Phi_1 + w_n$, with $w_n \in \operatorname{span}[\Phi_1]^\perp$, from Proposition 5.4 (i), we have $\|w_n\|_{L^\infty(\partial\Omega)} = O(|s_n|^\alpha)$.

Now, from hypotheses (H1)-(H2), Lemma 2.13, hypotheses (103), and Lemma 5.5, we get that

$$\begin{aligned}
\left(\int_{\partial\Omega} \Phi_1^2\right) \liminf_{n\to\infty} \frac{\sigma_1 - \lambda_n}{\|u_n\|_{L^\infty(\partial\Omega)}^{\alpha-1}} \quad &\geq \quad \liminf_{n\to\infty} \int_{\partial\Omega} \frac{(s_n\Phi_1 + w_n) g(\cdot, s_n\Phi_1 + w_n)}{|s_n\Phi_1 + w_n|^{1+\alpha}} \Phi_1^{1+\alpha} \\
&\geq \quad \liminf_{n\to+\infty} \int_{\partial\Omega} \frac{s_n g(\cdot, s_n\Phi_1)}{|s_n|^{1+\alpha}} \Phi_1 \ > 0,
\end{aligned} \tag{105}$$

and therefore $\lambda_n < \sigma_1$.

Analogously, for (λ_n', u_n') we get $\lambda_n' > \sigma_1$. Hence (i) is proved.

To prove (ii), assume, by choosing subsequences if necessary, that $s_n < s_n' < s_{n+1}$ for all $n \geq 0$ and that $s_n, s_n' \geq S_0$, where S_0 is the one from Proposition 5.4 (ii). In particular, from (i) and (ii) of Proposition 5.4 we have that if $(\lambda, u) \in \mathcal{D}^+$ and $P(u) = s > S_0$, then

$$\|u\|_{L^\infty(\partial\Omega)} \leq (1 + C_1 \|G_1\|_{L^r(\partial\Omega)} |S_0|^{\alpha-1}) s.$$

Again, taking S_0 large enough we can assume $\|u\|_{L^\infty(\partial\Omega)} \leq 2s$.

Define the set

$$K_n = \{(\lambda, u) \in \mathcal{D}^+ : P(u) = s \,\text{and}\, s_n \leq s \leq s_{n+1}\}. \tag{106}$$

Let us show that, for each $n \in \mathbb{N}$, K_n is a compact set in $\mathbb{R} \times C(\bar{\Omega})$.

Let us take a sequence in $(\mu_k, v_k) \in K_n$, and let us extract a subsequence, that we also denote by (μ_k, v_k) with the property that $\mu_k \to \mu^*$. Obviously $s_n \leq P(v_k) \leq s_{n+1}$ for all k, which implies the bounds $\|v_k\|_{C(\partial\Omega)} \leq 2s_{n+1}$ for all k.

Using these a priori bounds on the solutions, we have (see [7, Proposition 2.3])

$$\|v_k\|_{C^\alpha(\bar{\Omega})} \leq C_1 \left(1 + \|v_k\|_{L^\infty(\partial\Omega)}\right) \leq C,$$

for some C independent of k. Using the compact embedding $C^\alpha(\bar{\Omega}) \hookrightarrow C^\beta(\bar{\Omega})$ for $0 < \beta < \alpha$, we obtain that there exists another subsequence, that we denote the same, and a function $u^* \in C^\beta(\bar{\Omega})$ such that $v_k \to u^*$ in $C^\beta(\bar{\Omega})$. Observe that v_k satisfies

$$\begin{cases} \Delta v_k + v_k &= 0, & \text{in } \Omega, \\ \dfrac{\partial v_k}{\partial n} &= \mu_k v_k + g(x, v_k), & \text{on } \partial\Omega, \end{cases}$$

and the regularity of g implies

$$g(\cdot, v_k) \to g(\cdot, u^*)$$

pointwise. Now, hypothesis (H2) and the Lebesgue dominated convergence Theorem imply that

$$g(\cdot, v_k) \to g(\cdot, u^*) \, \text{in} \, L^r(\partial\Omega)$$

as $k \to \infty$.

Passing to the limit in the weak formulation of the above equation, we get that u^* is a solution of

$$\begin{cases} -\Delta u^* + u^* &= 0, & \text{in } \Omega, \\ \dfrac{\partial u^*}{\partial n} &= \mu^* u^* + g(x, u^*), & \text{on } \partial\Omega, \end{cases}$$

while the convergence of v_k implies $s_n \leq s^* = P(u^*) \leq s_{n+1}$. Hence, $(\mu^*, u^*) \in K_n$. This shows the compactness of K_n.

Observe that since $s_n < s'_n < s_{n+1}$, there exists $(\lambda, u) \in K_n$ with $\lambda > \sigma_1$. Hence, if we define the number

$$\lambda_n^* = \sup\{\lambda : (\lambda, u) \in K_n\}, \tag{107}$$

then $\lambda_n^* > \sigma_1$, and from the compactness of K_n there exists u_n^* such that $(\lambda_n^*, u_n^*) \in K_n$. From (i) and the fact that $\lambda_n^* > \sigma_1$, we have that

$$s_n < P(u_n^*) < s_{n+1}.$$

But this implies that there is no solution (λ, u) nearby (λ_n^*, u_n^*) with $\lambda > \lambda_n^*$. If this were the case, then by continuity of the projection P we would have for such a solution $s_n < P(u) < s_{n+1}$, so that $(\lambda, u) \in K_n$ and therefore λ_n^* would not satisfy (107). Hence, (λ_n^*, u_n^*) is a supercritical turning point.

With a completely symmetric argument, using the sets

$$K'_n = \{(\lambda, u) \in \mathcal{D}^+, : \, P(u) = s, \quad s'_n \leq s \leq s'_{n+1}\},$$

and defining $\lambda_{*,n} = \inf\{\lambda : (\lambda, u) \in K'_n\}$, we show the existence of u_* such that $(\lambda_{*,n}, u_{*,n}) \in K'_n$ is a subcritical turning point.

In order to prove the existence of resonant solutions, let us show now the following: there exists $n_0 \in \mathbb{N}$ large enough such that for each $n \geq n_0$ both sets K_n and K'_n contain resonant solutions, that is, solutions of the form (σ_1, u).

Let us provide the argument for the sets K_n. If this is not the case, then there will exist a sequence of integers numbers $n_j \to +\infty$ such that K_{n_j} does not contain any resonant solutions. This implies that the compact sets

$$K_{n_j}^+ = \{(\lambda, u) \in K_{n_j} : \lambda \geq \sigma_1\},$$

can be written as $K^+_{n_j} = \mathcal{D}^+ \cap \{(\lambda, u) \in \mathbb{R} \times C(\partial\Omega) : \lambda > \sigma_1, s_{n_j} < P(u) < s_{n_j+1}\}$, and therefore $K^+_{n_j}$ contains at least a connected component of $\mathcal{D}^+$. Moreover, it is nonempty, since we know that there exists at least one solution (λ, u) with $P(u) = s'_{n_j} \in (s_{n_j}, s_{n_j+1})$, and therefore $\lambda > \sigma_1$. The fact that we can construct a sequence of connected componets of $\mathcal{D}^+$ contradicts the fact that $\mathcal{D}^+$ is a continuum near $(\sigma_1, +\infty) \in \mathbb{R} \times C(\bar{\Omega})$.

A completely symmetric argument can be applied to the sets K'_n. ☑

With the tools above we can prove now the following.

Corollary 5.7. *With the definition of K_n and λ^*_n as in the proof of the theorem above (see* (106) *and* (107)*) we can show that, for n large enough,*

$$\{\lambda : \lambda \geq \sigma_1 \textit{ and } \exists u \textit{ with } (\lambda, u) \in K_n\} = [\sigma_1, \lambda^*_n].$$

Similarly, with the definition of K'_n and $\lambda_{*,n}$, we have

$$\{\lambda : \lambda \leq \sigma_1 \textit{ and } \exists u \textit{ with } (\lambda, u) \in K'_n\} = [\lambda_{*,n}, \sigma_1].$$

Proof. Assume the first statement is not true. This means that there exists a sequence of $n_j \to +\infty$ and a number $\tilde{\lambda}_{n_j} \in [\sigma_1, \lambda^*_{n_j}]$ such that there is no function $u \in C(\bar{\Omega})$ with $(\tilde{\lambda}_{n_j}, u) \in K_{n_j}$. Since we know that $(\lambda^*_{n_j}, u^*_{n_j}) \in K_{n_j}$, then necessarily $\sigma_1 \leq \tilde{\lambda}_{n_j} < \lambda^*_{n_j}$. Defining now

$$\tilde{K}_{n_j} = \{(\lambda, u) \in K_{n_j}, \lambda > \tilde{\lambda}_{n_j}\},$$

then $\tilde{K}_{n_j} \neq \emptyset$ since $(\lambda^*_{n_j}, u^*_{n_j}) \in K_{n_j}$ and with a similar argument as in the proof of the theorem above, we may show that K_{n_j} contains at least a nonempty connected component of $\mathcal{D}^+$. The fact that this can be obtained for the whole sequence $n_j \to +\infty$ is in contradiction with the fact that $\mathcal{D}^+$ is a continuum near $(\sigma_1, +\infty) \in \mathbb{R} \times C(\bar{\Omega})$.

A symmetric argument will show the second statement. ☑

5.12. Two examples

An oscillatory nonlinearity

Let us consider an oscillatory nonlinearity of the type (78), that is,

$$g(x, s) := s^\alpha \left[\sin\left(\left|\frac{s}{\Phi_1(x)}\right|^\beta\right) + C\right] \quad \text{with } \alpha < 1. \tag{108}$$

Applying Theorem 2.14 on subcritical and supercritical bifurcation, we have that if $\beta \in \mathbb{R}$ and $C > 1$, or if $\beta \leq 0$ and $C > 0$, then $\underline{\mathbf{G}}_+ > 0$, and the bifurcation from infinity is subcritical (see (25) for a definition of $\underline{\mathbf{G}}_+$).

On the other hand, if $\beta \in \mathbb{R}$ and $C < -1$, or if $\beta \leq 0$ and $C < 0$, then $\overline{\mathbf{G}}_+ > 0$, and the bifurcation from infinity is supercritical.

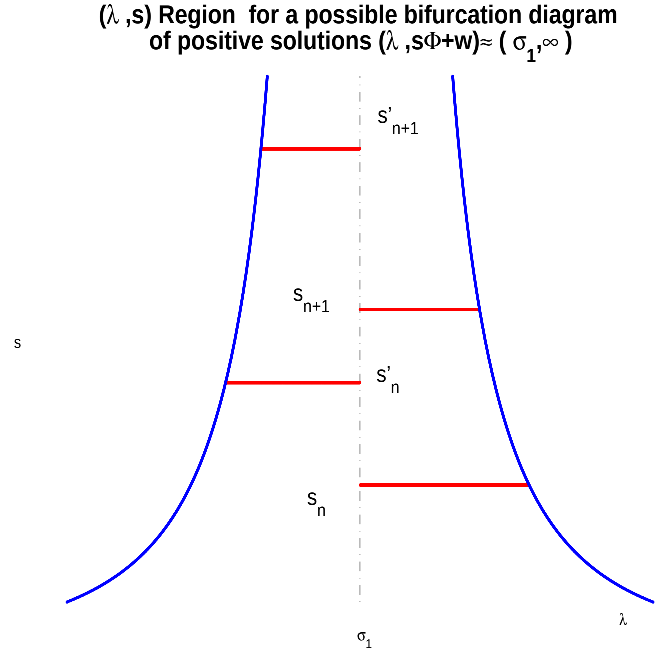


Figure 1. The (λ, s) region in $\mathbb{R}^2$ for a possible bifurcation diagram is the interior of the solid lines. From part (i) of the Theorem 5.6, the unbounded branch can not cross the solid horizontal lines.

Therefore, we consider here the range $\beta > 0$ and $-1 < C < 1$, and note that Theorem 5.6 applies if

$$\beta > 0, \qquad \alpha + \beta < 1, \qquad \text{and} \qquad -1 < C < 1.$$

Therefore, in this range of parameters, there exist unbounded sequences of subcritical and supercritical solutions, subcritical and supercritical turning points and infinite resonant solutions.

See Fig. 2 to visualize the parameter region and a bifurcation diagram.

Remark 5.8. The restriction $\alpha + \beta < 1$ on the size of β is needed in order to satisfy condition (102). This restriction means that although we need "oscillating" nonlinearities g, the oscillations cannot be very fast.

Although in principle the condition $\alpha+\beta < 1$ seems like a technical one (as it is suggested by the analysis of the one dimensional problem of the next section), it may be possible that for higher dimensional problems, some kind of homogenization phenomena may take place for very high oscillating nonlinearities that prevent the formation of turning points and/or resonant solutions.

This is also suggested by [19], where they show that with $\alpha = 0$, $\beta = 1$, if $1 \leq N \leq 5$ there are infinitely many solutions, while for $N \geq 6$ there are just a finite number of them.

Remark 5.9. We can also consider more general oscillatory nonlinearities of the type

$$g(x,s) := g_1(\lambda,x)s^\alpha \left[\sin\left(\left|\frac{s}{\Phi_1(x)}\right|^\beta\right) + g_2(\lambda,x)\right], \tag{109}$$

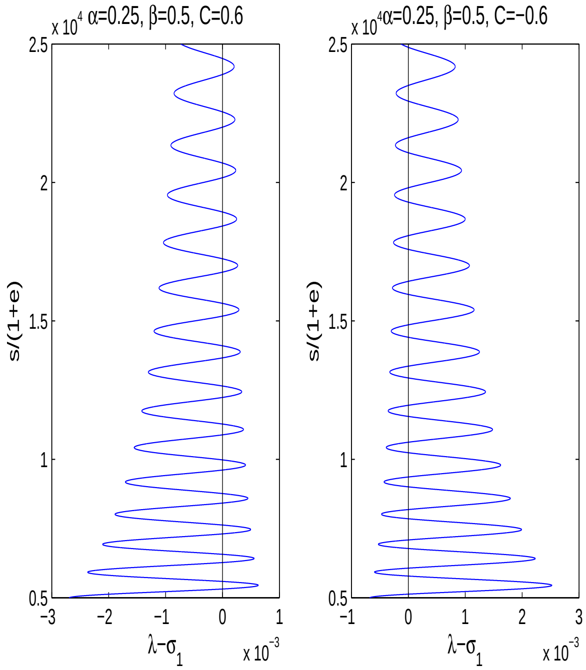


Figure 2. A bifurcation diagram of subcritical and supercritical solutions, containing infinite turning points and infinite resonant solutions.

where $g_1, g_2 \in C(\mathbb{R} \times \partial\Omega)$, $g_1(\lambda, x) \geq 0$, $g_1 \not\equiv 0$, and $-1 < C_1 \leq g_2(\lambda, x) \leq C_2 < 1$, and also for the same range of parameters $\alpha < 1$, $\beta > 0$, $\alpha + \beta < 1$.

An oscillatory branch for the one dimensional case

Now we consider the onedimensional version of (2), given by (35) for a particular g, where most computations can be made explicit (see Subsection 2.4).

Fix now

$$g(s) = s^\alpha \ \sin(s^\beta) \qquad \text{for any } \alpha < 1,\ \beta > 0.$$

From definition (25) we can write

$$\begin{aligned}
\underline{\mathbf{G}}_+ \quad &:= \quad \int_{\partial\Omega} \liminf_{s\to+\infty} \frac{sg(s)}{|s|^{1+\alpha}}\ \Phi^{1+\alpha} = \int_{\partial\Omega} \liminf_{s\to+\infty}\ \sin(s^\beta)\ \Phi^{1+\alpha} = -\int_{\partial\Omega} \Phi^{1+\alpha} < 0,\\
\overline{\mathbf{G}}_+ \quad &:= \quad \int_{\partial\Omega} \limsup_{s\to+\infty} \frac{sg(s)}{|s|^{1+\alpha}}\ \Phi^{1+\alpha} = \int_{\partial\Omega} \limsup_{s\to+\infty}\ \sin(s^\beta)\ \Phi^{1+\alpha} = \int_{\partial\Omega} \Phi^{1+\alpha} > 0,
\end{aligned}$$

and then $\underline{\mathbf{G}}_+ < 0 < \overline{\mathbf{G}}_+$.

Moreover, by looking in (37) at the values of $r \in \mathbb{R}$ such that $\lambda(r) = \sigma_1$, we get that (σ_1, u_k) is a solution for any $k \in \mathbb{Z}$, where

$$u_k(x) := \frac{(k\pi)^{1/\beta}}{e+1}(e^x + e^{1-x}),$$

i.e., there is an unbounded sequence of solutions of the resonant problem (see Fig. 3).

Moreover, computing in (37) the local maxima and minima of $\lambda(r)$, we get that (λ_k^*, u_k^*) is an unbounded sequence of turning points, where

$$\lambda_k^* := \sigma_1 - \frac{(-1)^k\, \alpha}{[(k+1/2)\pi]^{1-\alpha}},$$

and

$$u_k^*(x) := \frac{[(2k+1)\pi]^{1/\beta}}{2(e+1)}(e^x + e^{1-x})$$

(see Fig. 3).

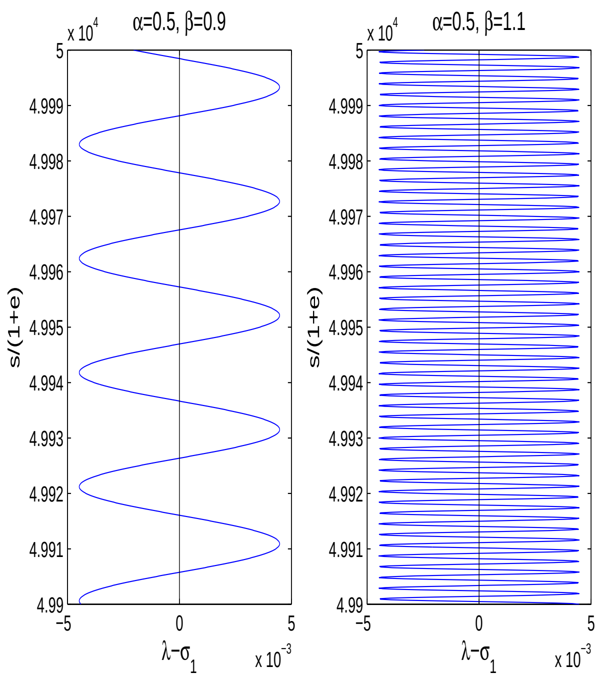


Figure 3. $\alpha = 0.5$, $\beta = 0.9$ and $\beta = 1.1$.

6. Stability switches

In this Section we consider solutions to the elliptic problem with nonlinear boundary conditions (2), assuming that g is sublinear at infinity and oscillatory. We provide sufficient conditions on g for the existence of unbounded sequences of stable solutions, unstable solutions, and turning points, even in the absence of resonant solutions. The main difference with Section 5 is that our arguments rely on proving stability switches, instead of in the cross of the principal eigenvalue. In fact, we provide sufficient conditions for having infinitely many stability switches in a subcritical branch, without no one resonant solution (see Theorem 6.1 and Theorem 6.5).

6.13. Introduction to stability switches

Throughout this Section we assume that the nonlinearity satisfies hypotheses (H1), (H2), (H3) and, moreover, we consider the following extra hypothesis.

(H4) The second partial derivative $g_{ss}(\lambda,\cdot,\cdot) \in C(\partial\Omega \times \mathbb{R})$ is such that

$$\sup_{|s|\geq M} \left\| \frac{g_{ss}(\cdot,s)}{|s|^{\rho-\alpha-1}} \right\|_{L^\infty(\partial\Omega)} \to 0 \qquad \text{as } M\to\infty \quad \text{and } \lambda\to\sigma_1. \tag{110}$$

Let $\{\sigma_i\}_{i=1}^\infty$ denote the sequence of *Steklov* eigenvalues of the problem 3. We recall that the Steklov eigenvalues form an increasing sequence of real numbers, $\{\sigma_i\}_{i=1}^\infty$. Each eigenvalue has finite multiplicity. The first eigenvalue σ_1 is simple and, due to Hopf's Lemma, we may assume its eigenfunction Φ_1 to be strictly positive in $\overline{\Omega}$. The eigenfunctions are orthogonal in $L^2(\partial\Omega)$, and we take $\|\Phi_1\|_{L^\infty(\partial\Omega)}=1$.

As stated in Theorem 2.10, due to (H2) there exists a connected set of positive solutions of (2). We denote it by $\mathcal{D}^+ \subset \mathbb{R}\times C(\bar\Omega)$, and recall that for $(\lambda,u_\lambda)\in\mathcal{D}^+$,

$$u=s\Phi_1+w, \qquad \text{with } \ w=o\left(|s|\right) \quad \text{and} \quad |\sigma_1-\lambda|=o(1) \quad \text{as } |s|\to\infty.$$

The set $\mathcal{D}^+$ is known as a *branch bifurcating from infinity* in the sense of Rabinowitz (cf. [31, 7]).

For $(\lambda,u_\lambda)\in\mathcal{D}^+$ we say that u_λ is a *stable* solution if there exists a neighborhood of u_λ in $C(\bar\Omega)$ such that, for initial data in that neighborhood, the solution to the parabolic problem

$$\left\{\begin{array}{rcll} u_t-\Delta u+u &=& 0, & \text{in } \Omega\times\mathbb{R}^+,\\ \dfrac{\partial u}{\partial n} &=& \lambda u+g(x,u), & \text{on } \partial\Omega\times\mathbb{R}^+,\\ u(0,x) &=& u_0(x), & \text{in } \Omega, \end{array}\right. \tag{111}$$

converges to u_λ as $t\to+\infty$. On the other hand, we say that u_λ is *unstable* if any neighborhood of u_λ contains initial conditions such the solution to (111) leaves that neighborhood in finite time. That is asymptotic stability in the Lyapunov sense.

Our goal is to give conditions on the sublinear oscillatory term g that guarantee the existence of unbounded sequences of stable solutions, unstable solutions and turning points (see Definition 1.1 of turning points).

Our main result, Theorem 6.5 below, is exemplified by the case in which

$$g(x,s):=s^\alpha\left[\sin\left(\left|\frac{s}{\Phi_1(x)}\right|^\beta\right)+C\right], \qquad \text{with } \alpha<1. \tag{112}$$

In fact, we have the following result (the proof of this Theorem follows directly from Theorem 6.5, we do not include it here, and leave it for the reader).

Theorem 6.1. *Assume that g is given by* (112)*. If*

$$\beta>0 \qquad \textit{and} \quad \alpha+\beta<1,$$

then the unbounded branch of positive solutions of (2) *contains a sequence of stable solutions, a sequence of unstable solutions and a sequence of turning points.*

In Figures 4 and 5 we plot the bifurcation diagram in the one dimensional case for g as above. Figure 6 sketches the changes of stability of solutions.

Remark 6.2. Assume that $C > 1$ in equation (112); then, the bifurcation is subcritical (see Theorem 2.14), on the other side, if $C < -1$, then the bifurcation is supercritical, and in any case there are not resonant solutions, and results on Section 5 does not apply.

Instead of that, changes of stability are the key in this situation. Let us point out that Theorem 6.1 apply for any $C \in \mathbb{R}$.

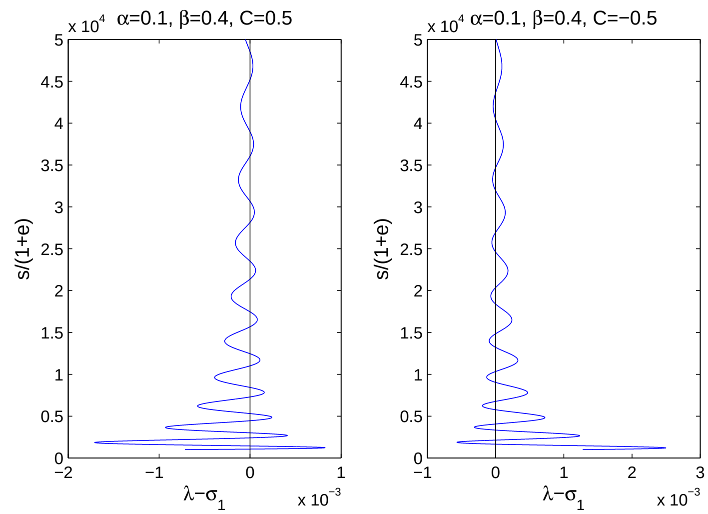


Figure 4. Bifurcation diagram having infinitely many sub-critical solutions ($\lambda < \sigma_1$), super-critical solutions ($\lambda > \sigma_1$), stable solutions, unstable solutions, turning points and resonant solutions ($\lambda = \sigma_1$).

Our result is sharp in the fact that, if condition (118) fails, all solutions in $\mathcal{D}^+$ may be either stable or unstable for s big enough (see Theorem 4.5). Our result proves the existence of infinitely many turning points, even in the absence of resonant solutions (see Figure 5). There it can be seen that the unbounded sequence of turning points given by Theorem 6.5 can be either *subcritical* (i.e., for values of the parameter $\lambda < \sigma_1$), see Figure 5 left, or *supercritical* (i.e., for for values of the parameter $\lambda > \sigma_1$), see Figure 5 right, or may have a sequence of subcritical solutions as well a sequence of supercritical solutions. Hence, by connectedness of $\mathcal{D}^+$, the branch contains infinitely many *resonant* solutions (i.e., for $\lambda = \sigma_1$), see Figure 4.

The main difference with Section 5 is the possibility of existence of a subcritical (or supercritical) branch. Precisely, the main ingredient for the proof of the existence of infinitely many turning points in Section 5 was the existence of infinitely many subcritical and supercritical solutions in a connected branch, and consequently of infinitely many resonant solutions.

Related results for the case of a nonlinear reaction in Ω and homogeneous Dirichlet boundary conditions were established in [10, 14, 19, 25]. In [19] the authors work in the unit ball $B \subset \mathbb{R}^N$ with $N \geq 1$, and the nonlinear term is $\lambda u + \sin(u)$. They proved

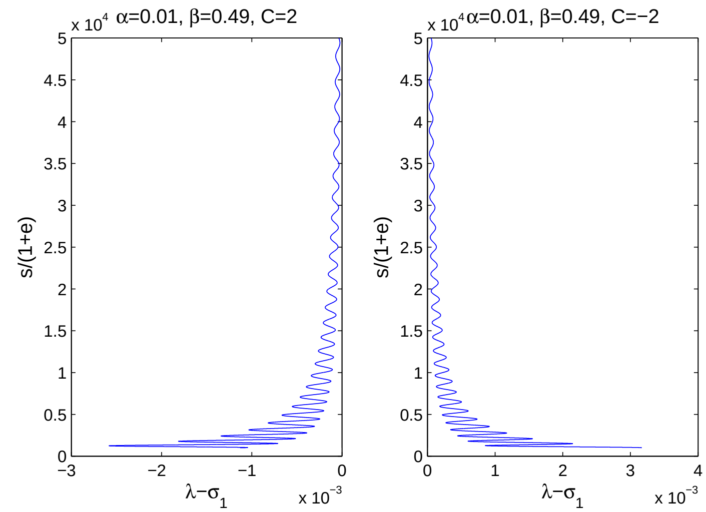


Figure 5. A bifurcation diagram of stable and unstable solutions: on the left all of them are subcritical, on the right all of them are supercritical, and none is resonant.

that when $\lambda = \lambda_1$, the first eigenvalue with Dirichlet boundary conditions, the problem has infinitely many solutions for $1 \le N \le 5$, and at most finitely many solutions for $N \ge 6$. Similar oscillatory phenomena, sometimes known as *snaking bifurcation*, can be observed in higher-order PDE (cf. [35] and [23]). We refer the reader to [20, 21] for related problems with nonlinear boundary conditions.

This section is organized as follows. In subsection 6.14 we collect some essentially known results on Lyapunov stability. Subsection 6.15 contains our main result, giving sufficient conditions for having stable and unstable solutions. Finally, subsection 6.16 presents two examples, the typical oscillatory nonlinearity (112) and the one dimensional case.

6.14. Lyapunov function and stability

For λ fixed we consider

$$I(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right) - \frac{\lambda}{2}\int_{\partial\Omega} u^2 - \int_{\partial\Omega} G(\cdot, u),$$

where $G(\lambda, x, s) := \int_{s_0}^{s} g(x,t)\,dt$ for some $s_0 \gg 1$ fixed. An elementary calculation shows that if u is a solution to the parabolic equation (111), then $\dfrac{d}{dt} I(u(t)) = I'(u(t))u_t \le 0$, i.e., I is Lyapunov function for the parabolic problem (111).

Moreover, if u_λ is a solution to (2), then it is a critical point for I. Furthermore, u_λ is stable if the quadratic form

$$Q_{u_\lambda}((v) = \int_\Omega |\nabla v|^2 + v^2 - \int \partial\Omega \lambda v^2 + g_s(\cdot, u_\lambda v^2) \tag{113}$$

is positive definite. On the other hand, if Q_{u_λ} is negative definite in one direction, then u_λ is unstable. Thus we have

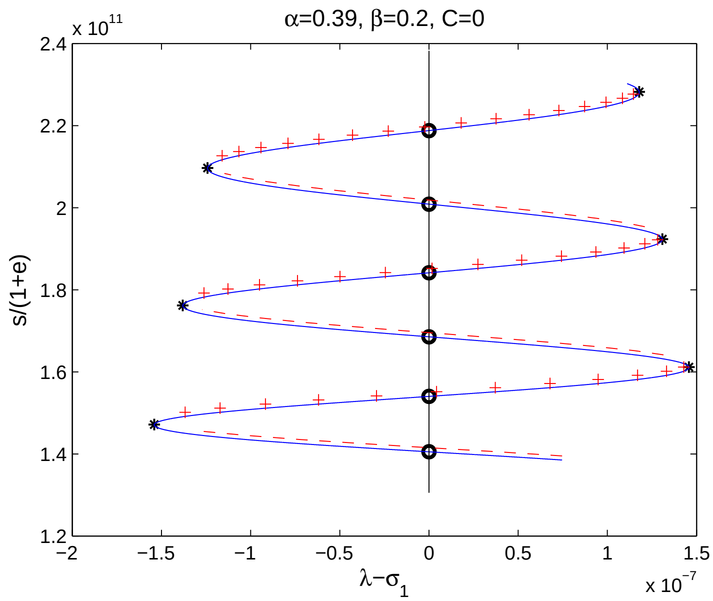


Figure 6. Bifurcation diagram and sketch of the stability of solutions, + for stable solutions and − for unstable solutions. The symbol $*$ marks turning points and $\circ$ resonant solutions.

Lemma 6.3. *If $\mu_1 \equiv \mu_1(\lambda, u_\lambda)$ denotes the principal eigenvalue of*

$$\begin{cases} -\Delta\varphi_1 + \varphi_1 &= 0, & \text{in } \Omega, \\ \dfrac{\partial \varphi_1}{\partial n} &= \mu_1\varphi_1 + g_s(\lambda, x, u_\lambda)\varphi_1, & \text{on } \partial\Omega, \end{cases} \tag{114}$$

then u_λ is stable if $\mu_1 > \lambda$. Also u_λ is unstable if $\mu_1 < \lambda$.

Proof. Suppose $\mu_1 > \lambda$. The variational characterization of μ_1 states that

$$\mu_1 := \inf_{u \in H^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 + u^2 - \int_{\partial\Omega} g_s(\lambda, \cdot, u_\lambda)u^2}{\int_{\partial\Omega} u^2}.$$

Therefore, for any $u \in H^1(\Omega) \setminus \{0\}$ we have

$$\begin{aligned} 0 &\le \int_\Omega |\nabla u|^2 + u^2 - \int_{\partial\Omega} \mu_1 u^2 + g_s(\lambda, \cdot, u_\lambda)u^2 \\ &< \int_\Omega |\nabla u|^2 + u^2 - \int_{\partial\Omega} \lambda u^2 + g_s(\lambda, \cdot, u_\lambda)u^2. \end{aligned}$$

Hence Q_{u_λ} is positive definite and u_λ is stable.

On the other hand, if $\mu_1 < \lambda$, letting φ_1 denote the eigenfunction corresponding to the eigenvalue μ_1, then

$$\begin{aligned} 0 &= \int_\Omega \|\nabla\varphi_1\|^2 + \varphi_1^2 - \int_{\partial\Omega} \mu_1\varphi_1^2 + g_s(\lambda,\cdot,u_\lambda)\varphi_1^2 \\ &> \int_\Omega \|\nabla\varphi_1\|^2 + \varphi_1^2 - \int_{\partial\Omega} \lambda\varphi_1^2 + g_s(\lambda,\cdot,u_\lambda)\varphi_1^2. \end{aligned}$$

Thus Q_{u_λ} is negative definite in the direction of φ_1, which proves that u_λ is unstable. ☑

6.15. Infinitely many stability switches

This section is devoted to giving sufficient conditions for the existence of unbounded sequences of stable solutions, unstable solutions, and turning points of (2).

Let α be the rate with which g goes to infinity (see (H2)), and ρ be rate with which $g - sg_s$ goes to infinity (see (H3)). In the first place we note that even if $\alpha \neq \rho$, then the *boundary Steklov eigenvalue* $\mu_1 \to \sigma_1$ and the *boundary Steklov eigenfunction* $\varphi_1 \to \Phi_1$ as $\lambda \to \sigma_1$ and $\|u\|_{L^\infty(\partial\Omega)} \to \infty$ (see Lemma 4.3).

Next, we analyze the changes of stability. To do that, we look at a detailed account of the asymptotic behavior of the nonlinear term

$$\underline{\mathbf{F}}_+ := \int_{\partial\Omega} \liminf_{(\lambda,s)\to(\sigma_1,+\infty)} \frac{sg(\cdot,s) - s^2 g_s(\cdot,s)}{|s|^{1+\rho}} \, \Phi_1^{1+\rho},$$

for $\rho < 1$. Changing lim inf by lim sup, we define the number $\overline{\mathbf{F}}_+$. If

$$\underline{\mathbf{F}}_+ > 0, \qquad \text{then } \mathcal{D}^+ \text{ is stable and subcritical}$$

(see Theorem 4.5), and if

$$\overline{\mathbf{F}}_+ < 0, \qquad \text{then } \mathcal{D}^+ \text{ is unstable and supercritical}$$

(see Theorem 4.6). In this Section we consider nonlinearities for which

$$\underline{\mathbf{F}}_+ < 0 < \overline{\mathbf{F}}_+.$$

Unlike the case $\underline{\mathbf{F}}_+ > 0$, or $\overline{\mathbf{F}}_+ < 0$, our assumption $\underline{\mathbf{F}}_+ < 0 < \overline{\mathbf{F}}_+$ allows for the existence of sequences of stable supercritical solutions and unstable subcritical solutions (see Theorem 6.5).

We shall argue as in Subsection 2.3 for the sub-critical and supercritical case. To determine whether a sequence of solutions (λ_n, u_n) is stable or unstable, one must check the sign of

$$\liminf_{n\to\infty} F(u_n) \qquad \text{and of} \quad \limsup_{n\to\infty} F(u_n), \tag{115}$$

where F is defined by (117). This is done in Lemma 4.4. But this requires an a priori knowledge of the solutions themselves, which is in general impracticable.

In Proposition 5.4, it is proved that when g is such that

$$|g(x,s)| = O\left(|s|^\alpha\right) \text{ as } |s| \to \infty \text{ for some } \alpha < 1,$$

then, the solutions in $\mathcal{D}^{\pm}$ can be described as

$$u_n = s_n\Phi_1 + w_n, \qquad \text{where} \quad \int_{\partial\Omega} w_n\Phi_1 = 0, \quad \text{and} \qquad w_n = O(|s_n|^{\alpha}) \text{ as } n \to \infty,$$

and we intend to unveil the signs in (115) by just looking at the signs of those lim inf at $\lambda_n = \sigma_1$ and $u_n = s_n\Phi_1$. This is achieved in Lemma 6.4.

With these tools, in Theorem 6.5 we take two sequences $\{s_n\}$ and $\{s_n'\}$ satisfying

$$-\infty < \lim_{n\to+\infty} F(s_n'\Phi_1) \; < 0 < \lim_{n\to+\infty} F(s_n\Phi_1) \; < \infty, \tag{116}$$

and from here we obtain the existence of unbounded sequences of stable and unstable solutions of (2) in $\mathcal{D}^+$.

We will use Lemma 4.4, that allow us to compare λ and μ_1 as $\lambda \to \sigma_1$.

In order to prove the main result, we have to guarantee that the signs in (115) can be deduced from those of (116). This is stated in the following technical result, which is Lemma 5.5 applied for $h = g - sg_s$.

Lemma 6.4. *Assume that g satisfies hypotheses* (H1)*,* (H2)*,* (H3) *and* (H4)*.*

If $\lambda_n \to \sigma_1$, $s_n \uparrow \infty$ and there exists a constant C such that $\|w_n\|_{L^\infty(\partial\Omega)} \leq C|s_n|^{\alpha}$ for all $n \to \infty$, then

$$\liminf_{n\to+\infty} \; F(s_n\Phi_1 + w_n) \geq \liminf_{n\to+\infty} \; F(s_n\Phi_1),$$

where F is given by (117)*. Similarly,*

$$\limsup_{n\to+\infty} \; F(s_n\Phi_1 + w_n) \leq \limsup_{n\to+\infty} \; F(s_n\Phi_1).$$

We are now in a position to prove our main result, which states the existence of unbounded sequences of stable solutions, unbounded sequences of unstable solutions and also unbounded sequences of turning points.

Our main result is the following.

Theorem 6.5. *Assume the nonlinearity g satisfies hypothesis* (H1)*,* (H2)*,* (H3) *and* (H4)*. Let $F : \mathbb{R} \times C(\bar{\Omega}) \to \mathbb{R}$ be defined by*

$$F(u) := \int_{\partial\Omega} \frac{ug(\cdot,u) - u^2 g_s(\cdot,u)}{|u|^{1+\rho}} \Phi_1^{1+\rho}. \tag{117}$$

If there exist sequences $\{s_n\}$, $\{s_n'\}$ converging to $+\infty$, such that

$$\lim_{n\to+\infty} F(s_n'\Phi_1) \; < 0 < \lim_{n\to+\infty} F(s_n\Phi_1), \tag{118}$$

then

(i) *There exists a sequence $\{(\lambda_n, u_n)\} \in \mathcal{D}^+$ of stable solutions to* (2) *and a sequence $\{(\lambda_n', u_n')\} \in \mathcal{D}^+$ of unstable solutions such that $(\lambda_n, \|u_n\|_{L^\infty(\partial\Omega)}) \to (\sigma_1, \infty)$ and $(\lambda_n', \|u_n'\|_{L^\infty(\partial\Omega)}) \to (\sigma_1, \infty)$ as $n \to \infty$.*

(ii) *There exists a sequence* $\{(\lambda_n^*, u_n^*)\} \in \mathcal{D}^+$ *of turning points such that* $(\lambda_n^*, \|u_n^*\|_{L^\infty(\partial\Omega)}) \to (\sigma_1, \infty)$ *as* $n \to \infty$.

Proof. (i) To prove the result, we show that from (118) we can find two unbounded sequences of solutions $\{(\lambda_n, u_n)\}$, $\{(\lambda_n', u_n')\}$, with λ_n, λ_n' close enough to σ_1, such that $\mu_{1,n} := \mu_1(\lambda_n, u_n) > \lambda_n$ and $\mu_{1,n}' := \mu_1(\lambda_n', u_n') < \lambda_n'$ respectively, and then we use Lemma 6.3. We below focus in the stable case and the unstable one is analogous.

Since the projection of the unbounded branch of positive solutions on span$[\Phi_1]$ is an interval $[s_0, \infty)$, choose $(\lambda_n, u_n) \to (\sigma_1, \infty)$ such that

$$P(u_n) := \frac{\int_{\partial\Omega} u_n \Phi_1}{\int_{\partial\Omega} \Phi_1^2} = s_n,$$

with s_n as in (118). Writing $u_n = s_n\Phi_1 + w_n$, from [9, Proposition 3.2] and hypotheses (H3), we obtain that $w_n = O(|s_n|^\alpha)$.

Taking into account Lemma 4.4 we have

$$\liminf_{n\to\infty} \frac{\mu_{1,n} - \lambda_n}{\|u_n\|_{L^\infty(\partial\Omega)}^{\rho-1}} \geq \liminf_{n\to\infty} \frac{\mu_{1,n} - \lambda_n}{\|u_n\|_{L^\infty(\partial\Omega)}^{\rho-1}} \geq \frac{1}{\int_{\partial\Omega} \Phi_1^2} \liminf_{n\to\infty} F(\lambda_n, u_n). \tag{119}$$

Applying Lemma 5.5 to the function $h = g - sg_s$, by hypothesis (H2) - (H4), and (118), we infer

$$\liminf_{n\to\infty} F(s_n\Phi_1 + w_n) \geq \liminf_{n\to+\infty} F(s_n\Phi_1) > 0. \tag{120}$$

The inequalities (119)-(120) imply that $\mu_{1,n} > \lambda_n$ for λ_n close enough to σ_1. Likewise, it can be proved that $\mu_{1,n}' < \lambda_n'$ for λ_n' close enough to σ_1, ending this part of the proof.

(ii) To achieve this part of the proof, we use Leray-Schauder degree theory. Let

$$K_n := \{(\lambda, u) \in \mathcal{D}^+ : P(u) = s \text{ and } s_n \leq s \leq s_n'\}.$$

For each $n \in \mathbb{N}$, K_n is a compact set in $\mathbb{R} \times C(\bar{\Omega})$ (see for instance [9, Proof of Theorem 3.4]). For each $n \in \mathbb{N}$ fix, let $\lambda_{min} := \min\{\lambda : (\lambda, u) \in K_n\}$, and likewise λ_{max}. Assume on the contrary that K_n contains no turning point. In other words, assume that for each $\lambda \in [\lambda_{min}, \lambda_{max}]$ there exist a unique solution $u_\lambda \in K_n$.

For any $b \in L^q(\partial\Omega)$, $q \geq 1$, there exists a unique solution of

$$\begin{cases} -\Delta v + v &= 0, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} &= b, & \text{on } \partial\Omega. \end{cases}$$

Moreover, $\|v\|_{W^{1,p}(\Omega)} \leq C\|b\|_{L^q(\partial\Omega)}$, with $p = q\frac{N}{N-1}$. We denote it by $T(b) = v$ and

$$S(b) := \gamma T(b), \qquad \text{where } \gamma : W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega) \text{ is the trace operator.}$$

The operator S is known as the *Neumann-to-Dirichlet operator*. If $q > N - 1$, then the map S transforms $L^q(\partial\Omega)$ into $C^\tau(\partial\Omega)$ for some $\tau \in (0, 1)$, and is continuous and compact (see for instance [7, Lemma 2.1]).

Let $H : [\lambda_{min}, \lambda_{max}] \times C(\partial\Omega) \to C(\partial\Omega)$ be the homotopy defined by

$$H(\lambda, u) := \lambda S u + S(g(\cdot, u)).$$

Hence, the fixed points of $H(\lambda, \cdot)$ are the solutions to (2). Let $\varepsilon > 0$; writing $u = s\Phi_1 + w$, and due to $\|w\|_{L^\infty(\partial\Omega)} = O(|s|^\alpha)$ with $\alpha < 1$, we obtain $\|u - s\Phi_1\|_{L^\infty(\partial\Omega)} \leq \varepsilon s$ for any s big enough.

Now consider the Leray-Schauder degree of $I - H(\lambda, \cdot)$ with respect to zero, in the set

$$\mathcal{O} := \bigcup_{s\in[s_n, s_n']} \{u \in C(\bar\Omega) : \|u - s\Phi_1\|_{L^\infty(\partial\Omega)} \leq 2\varepsilon s\}.$$

From the homotopy invariance property, $\deg_{LS}(I - H(\lambda, \cdot), \mathcal{O}, 0)$ is well defined and independent of λ for $\lambda \in [\lambda_{min}, \lambda_{max}]$. In particular,

$$\deg_{LS}(I - H(\lambda_n, \cdot), \mathcal{O}, 0) = \deg_{LS}(I - H(\lambda_n', \cdot), \mathcal{O}, 0). \tag{121}$$

Since from part (i) $\lambda_n < \mu_{1,n}$, the linearized operator $I - \lambda_n S - S[g_s(x, u_n)\cdot]$ is invertible and consequently u_n is an isolated fixed point. Therefore the fixed point index is well defined, and moreover

$$\mathrm{i}\ (H(\lambda_n, \cdot), u_n) = \deg_{LS}(I - \lambda_n S - S[g_s(x, u_n)\cdot], \mathcal{O}, 0) = (-1)^{m(\lambda_n)} = 1,$$

where $m(\lambda_n)$ is the sum of the algebraic multiplicities of the eigenvalues of the linearization strictly smaller than λ_n, and $m(\lambda_n) = 0$ if the linearization has no eigenvalues $\mu_{i,n}$ of this kind.

Moreover, from hypothesis u_n is the only solution in K_n for the value of the parameter $\lambda = \lambda_n$, so we deduce $\deg_{LS}(I - H(\lambda_n, \cdot), \mathcal{O}, 0) = \mathrm{i}\ (H(\lambda_n, \cdot), u_n)$.

On the other side,

$$\mathrm{i}\ (H(\lambda_n', \cdot), u_n') = \deg_{LS}(I - \lambda_n' S - S[g_s(x, u_n)\cdot), \mathcal{O}, 0] = -1,$$

and likewise $\deg_{LS}(I - H(\lambda_n', \cdot), \mathcal{O}, 0) = \mathrm{i}\ (H(\lambda_n', \cdot), u_n') = -1$, which contradicts (121) and the proof is accomplished. ☑

6.16. Two examples

The oscillatory nonlinearity

We summarize some known results for the nonlinearity (112). In Section 2 it is proved that if $\alpha < 1$, for any $\beta \in \mathbb{R}$, and $C \in \mathbb{R}$, there is an unbounded branch of positive solutions (see Theorem 2.7). Assume from now in advance that $\beta > 0$. In Theorem 2.14 it is proved that if $C > 1$, then the bifurcation is subcritical, while if $C < -1$, the bifurcation is supercritical, and in any case there are no resonant solutions (see Figure 5). In Theorem 5.6 it is proved that if $\beta > 0$, $\alpha + \beta < 1$, and $|C| < 1$, there exist unbounded sequences of subcritical and supercritical solutions, subcritical and supercritical turning points and infinite resonant solutions (see Figure 4). Case $|C| = 1$ is a critical case. In this particular example, if $|C| = 1$ we have an infinite sequence of resonant solutions given by

$$u_k(x) := [(2k \pm 1/2)\pi]^{1/\beta} \Phi_1(x), \qquad k \geq 0.$$

In this section we proved that if

$$\beta > 0, \qquad \alpha + \beta < 1, \qquad \text{and} \qquad \forall C \in \mathbb{R},$$

then the unbounded branch of positive solutions contains a sequence of stable solutions, a sequence of unstable solutions and a sequence of turning points (see Theorems 6.1 and 6.5).

Note that if $\alpha + \beta \geq 1$, then $g_s \not\to 0$ as $s \to \infty$, and therefore the eigenvalue of the linearized equation does not converge to the first boundary Steklov eigenvalue, i.e., $\mu_n \not\to \sigma_1$ as $n \to \infty$ (see Lemma 4.3 for $\alpha + \beta < 1$). In addition, condition (H4) in Theorem 6.5 cannot be satisfied, and stability of the solutions cannot be deduced from the signs on multiples of the eigenfunction (see the arguments explained at the beginning of Subsection 6.15 and also Lemma 5.5). Thus, the restriction $\alpha + \beta < 1$ is needed to guarantee both, for the convergence of eigenvalues and eigenvectors to σ_1 and Φ_1, respectively, and for hypothesis (H4) to be satisfied.

An example for the case $N = 1$

We make explicit some ideas on the one dimensional case for the problem (35). We know that the bifurcation problem is a two parameter nonlinear problem that can be treated using finite dimensional techniques (see (36)).

Choose $g(x, s) = s^{\alpha} \ \sin(s^{\beta})$ for any $\alpha < 1, \ \beta > 0$ (see Fig 7).

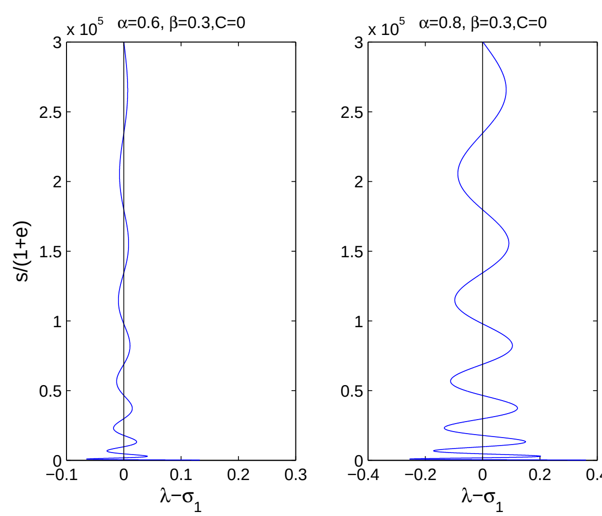


Figure 7. A bifurcation diagram of changing stability solutions; on the left $\alpha + \beta < 1$, on the right $\alpha + \beta > 1$, and in both cases $\lambda \to \sigma_1$.

The eigenvalue of the linearized equation is

$$\mu_1\big(-g_s(\lambda(s),\cdot,u_s)\big) := \frac{e-1}{e+1} - \alpha\,\frac{\sin\big[[s(e+1)]^\beta\big]}{[s(e+1)]^{1-\alpha}} - [s(e+1)]^{\alpha+\beta-1}\cos\big[[s(e+1)]^\beta\big].$$

If

$$[s(e+1)]^\beta = \begin{cases} (2k+1)\pi \\ \dfrac{(2k+1)\pi}{2} \\ 2k\pi \end{cases}, \qquad \text{then} \quad \mu_1\big(\lambda(s),u_s\big) - \lambda(s) \begin{cases} >0, \\ =0, \\ <0, \end{cases}$$

and we can conclude that (σ_1, u_{2k+1}), where

$$u_{2k+1}(x) := \frac{[(2k+1)\pi]^{1/\beta}}{e+1}(e^x + e^{1-x}) \qquad \text{for any} \quad k\in\mathbb{Z},$$

is a stable solution. Likewise, (σ_1, u_{2k}) is a sequence of unstable solutions, where

$$u_{2k}(x) := \frac{(2k\pi)^{1/\beta}}{e+1}(e^x + e^{1-x}) \qquad \text{for any} \quad k\in\mathbb{Z}.$$

Moreover, (λ_k^*, u_k^*) is an unbounded sequence of turning points, where

$$\lambda_k^* := \frac{e-1}{e+1} - \frac{(-1)^k\,\alpha}{[(k+1/2)\pi]^{1-\alpha}}, \qquad u_k^*(x) := \frac{[(2k+1)\pi]^{1/\beta}}{2(e+1)}(e^x+e^{1-x}).$$

The bifurcated branch from infinity contains stable and unstable solutions, and there is an unbounded sequence of turning points. See Figures 4, 5 and 3 for a bifurcation diagram when $N=1$. In that case, there is not restriction on the size of β (see Fig. 7).

We notice that with respect to the linearization, the things are different depending on $\alpha+\beta$. Note that if $\alpha+\beta\ge 1$ then $\mu_1\big(\lambda(s),\cdot,u_s\big) \nrightarrow \sigma_1$ as $s\to\infty$. On the other hand, the eigenvalue of the linearized equation satisfies $\mu_1\big(\lambda(s),\cdot,u_s\big) \to \sigma_1$ as $s\to\infty$, whenever $\alpha+\beta<1$ (see Fig. 8).

Moreover, if $\alpha+\beta<1$,

$$\begin{aligned}
\underline{\mathbf{F}}_+ &:= \int_{\partial\Omega} \liminf_{s\to+\infty} \frac{sg - s^2 g_s}{|s|^{1+\alpha+\beta}}\Phi^{1+\alpha+\beta} \\
&= \int_{\partial\Omega} \liminf_{s\to+\infty} \ -\beta\cos(s^\beta)\ \Phi^{1+\alpha+\beta} = -\beta\int_{\partial\Omega}\Phi^{1+\alpha+\beta}, \\
\overline{\mathbf{F}}_+ &:= \int_{\partial\Omega} \limsup_{s\to+\infty} \frac{sg - s^2 g_s}{|s|^{1+\alpha+\beta}}\Phi^{1+\alpha+\beta} \\
&= \int_{\partial\Omega} \limsup_{s\to+\infty} \ -\beta\cos(s^\beta)\ \Phi^{1+\alpha+\beta} = \beta\int_{\partial\Omega}\Phi^{1+\alpha+\beta},
\end{aligned}$$

i.e., $\underline{\mathbf{F}}_+ < 0 < \overline{\mathbf{F}}_+$.

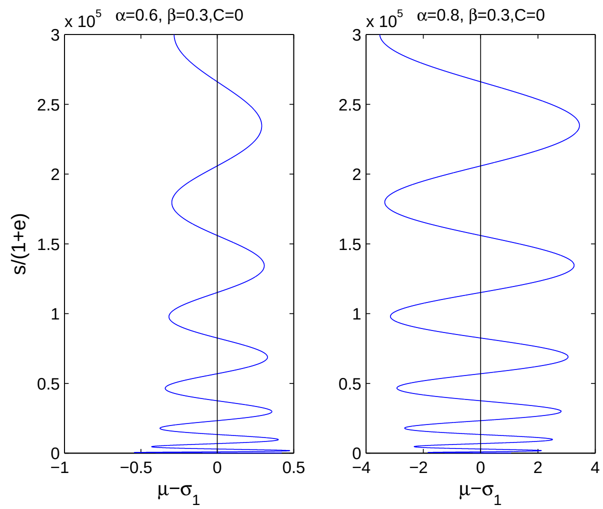


Figure 8. The difference between $\mu - \sigma_1$. On the left $\alpha + \beta < 1$, and $\mu \to \sigma_1$; on the right $\alpha + \beta > 1$, and $\mu \not\to \sigma_1$.

7. Bifurcation from zero

In this section we will consider the case where bifurcations from the trivial solution may occur (cf. [15, 12]). For this, we will need to assume that the nonlinearity g is $g(x,u) = o(u)$ as $u \to 0$.

We consider problem (2), but, instead of specifying the behavior of the nonlinearity g for large values of u, we consider the behavior of g for small values of u. That is, we assume

(H5) $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathèodory function (i.e., $g = g(x,s)$ is measurable in $x \in \Omega$, and continuous with respect to $(\lambda, s) \in \mathbb{R} \times \mathbb{R}$). Moreover, there exist $G_1 \in L^r(\partial\Omega)$ with $r > N - 1$ and continuous functions $\Lambda : \mathbb{R} \to \mathbb{R}^+$, and $U : \mathbb{R} \to \mathbb{R}^+$, satisfying

$$\begin{cases} \| g(x,s)| \le G_1(x) U(s), & \forall (x,s) \in \mathbb{R} \times \partial\Omega \times \mathbb{R}, \\ \lim\limits_{|s| \to 0} \dfrac{U(s)}{s} = 0, & \end{cases}$$

which in turn it implies that

$$\limsup_{|s| \to 0} \left| \frac{g(x,s)}{s} \right| \to 0,$$

that is, the function g is sublinear at 0 in the variable s.

We have the following result.

Theorem 7.1. *Consider problem* (2) *and assume that the nonlinearity g satisfies condition* (H5)*. If σ is an Steklov eigenvalue of odd multiplicity, then the set of solutions of*

(2) *possesses a component emanating from the bifurcation point* $(\sigma, 0) \in \mathbb{R} \times C(\bar{\Omega})$. *Moreover, this component, either it is bounded in* $\mathbb{R} \times C(\bar{\Omega})$, *in which case it meets another bifurcation point from zero (that is, another point* $(\sigma', 0)$ *for another Steklov eigenvalue* σ'*), or it is unbounded.*

Proof. The proof of this result follows the general results on bifurcations from the trivial solution given in [30]. See also [3] for similar results when the nonlinearity is in the interior. ☑

Remark 7.2. Observe that it is possible to have nonlinearities where both situations, the one from Theorem 7.1 and that from Theorem 2.10, hold. This is the case, for instance, where the nonlinearity $g(x,u)$ is $o(u)$ at $u \to 0$ and at $u \to \infty$. In this situation, both Theorems apply and if σ is an Steklov eigenvalue of odd multiplicity (for instance the first one) then both bifurcations, from zero and from infinity, occurs at this value of the parameter.

7.17. Resonant solutions and turning points accumulating to zero

We consider the elliptic equation (2), where now the nonlinear term $\dfrac{g(x,s)}{s} \to 0$ as $|s| \to 0$, and g is oscillatory. We provide sufficient conditions on g for the existence of sequences of resonant solutions and turning points accumulating to zero. A typical example of such a g is

$$g(x,s) := s^{\alpha}\left[\sin\left(\left|\frac{s}{\Phi_1(x)}\right|^{\beta}\right) + C\right] \qquad \text{with } \alpha+\beta > 1, \quad \beta < 0, \tag{122}$$

where Φ_1 stands for the first eigenfunction of the Steklov eigenvalue problem (3). The first eigenvalue σ_1 is simple and, due to Hopf's Lemma, we may assume its eigenfunction Φ_1 to be strictly positive in $\overline{\Omega}$, and we take $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$.

While in [9], [11], the case $\alpha+\beta<1$, $\beta>0$ is treated, we focus now on $\alpha+\beta>1$, $\beta<0$, inside of the complementary range. The case with $\alpha < 1$ corresponds to a *bifurcation from infinity* phenomenon (see Theorem 2.10, and also [7, 8, 9, 11, 31]). On the contrary, the case with $\alpha > 1$ corresponds to a *bifurcation from zero* phenomenon (see Theorem 7.1, and also [7, 15, 30]).

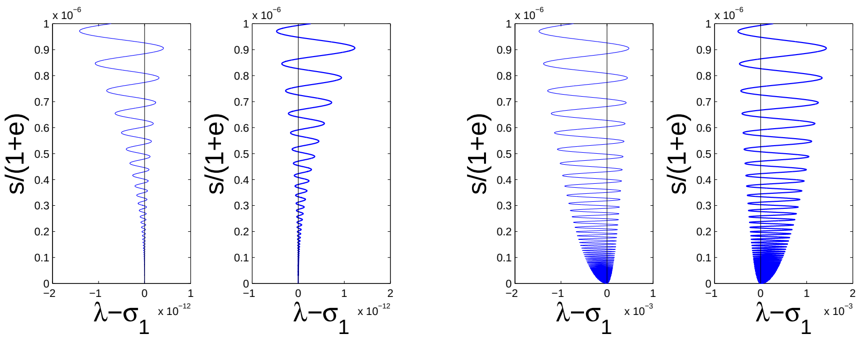
The oscillatory situation is in principle more complex than the monotone one, since order techniques such as sub and supersolutions are not applicable.

We perform an analysis of the local bifurcation diagram of non-negative solutions to (2), which turns out to be different from the case $\alpha < 1$ (see Figure 9 for $\alpha > 1$, and Figure 10 for $\alpha < 1$, and observe the different scales).

Throughout this Section we assume, besides (H5), that the following hypothesis hold

(H6)

$$\limsup_{|s|\to 0} \frac{U(s)}{|s|^{\alpha}} < +\infty \qquad \text{for some} \quad \alpha > 1.$$



$C = 0.5$ $\qquad$ $C = -0.5$ $\qquad$ $C = 0.5$ $\qquad$ $C = -0.5$

$\alpha = 3$ $\qquad$ $\alpha = 1.5$

Figure 9. Bifurcation diagram of subcritical and supercritical solutions, containing infinitely many turning points, and infinitely many resonant solutions. In all cases, $\beta = -0.35$.

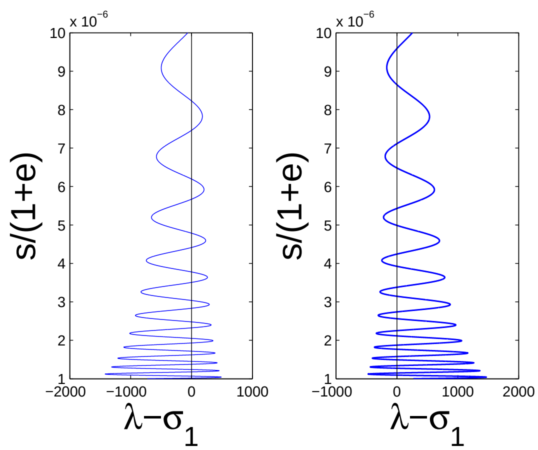


Figure 10. $\alpha = 0.5$.

(H7) The partial derivative $g_s(\cdot,\cdot) \in C(\partial\Omega \times \mathbb{R})$ (where $g_s := \dfrac{\partial g}{\partial s}$), $g_s(\cdot,\cdot,0) = 0$, and there exist $F_1 \in L^r(\partial\Omega)$, with $r > N-1$, and $\rho > 1$, such that

$$\frac{|g(x,s) - s g_s(x,s)|}{|s|^\rho} \le F_1(x) \qquad \text{as } \lambda \to \sigma_1,$$

for $x \in \partial\Omega$ and $s \le \varepsilon$ small enough.

Throughout this section, by solutions to (1.1) we mean elements $u \in H^1(\Omega)$ such that the weak formulation (10) holds. As proven in Proposition 2.3, all such solutions are in the Hölder space $C^\beta(\bar{\Omega})$, for some $\beta > 0$. Moreover, there exists a connected set of positive solutions of (2) known as a *branch bifurcating from zero* (cf. Theorem 7.1). We denote it by $\mathcal{C}^+ \subset \mathbb{R} \times C(\bar{\Omega})$, and recall that for $(\lambda, u_\lambda) \in \mathcal{C}^+$

$$u = s\Phi_1 + w, \qquad \text{with} \;\; w = o\left(|s|\right) \quad \text{and} \quad |\sigma_1 - \lambda| = o(1) \quad \text{as } |s| \to 0.$$

Our goal is to give conditions on the nonlinear oscillatory term g that guarantee the existence of sequences accumulating to zero of *subcritical* solutions (i.e., for values of the parameter $\lambda < \sigma_1$), *supercritical* solutions (i.e., for $\lambda > \sigma_1$), *resonant* solutions (i.e., for $\lambda = \sigma_1$), and turning points (see Definition 1.1 for a definition of turning point).

Our main result, Theorem 7.4 below, is exemplified by the case in which g is given by (122). In fact, we have:

Theorem 7.3. *Assume that g is given by* (122) *with $\beta < 0$. If*

$$|C| < 1, \qquad \text{and} \quad \alpha + \beta > 1,$$

then in any neighborhood of the bifurcation point $(\sigma_1, 0)$ in $\mathbb{R} \times C(\bar{\Omega})$, the branch $\mathcal{C}^+$ of positive solutions of (2) *contains a sequence of subcritical solutions, a sequence of supercritical solutions, a sequence of turning points, and a sequence of resonant solutions.*

The proof of this Theorem follows directly from Theorem 7.4.

Theorem 7.4. *Assume the nonlinearity g satisfies hypothesis* (H5), (H6) *and* (H7). *Let $G : \mathbb{R} \times C(\bar{\Omega}) \to \mathbb{R}$ be defined by*

$$G(u) := \int_{\partial\Omega} \frac{ug(\cdot, u)}{|u|^{1+\alpha}} \Phi_1^{1+\alpha}. \tag{123}$$

If there exist sequences $\{s_n\}$, $\{s'_n\}$, converging to 0^+, such that

$$\lim_{n\to+\infty} G(s'_n\Phi_1) < 0 < \lim_{n\to+\infty} G(s_n\Phi_1), \tag{124}$$

then

(i) For sufficiently large $n \gg 1$, if (λ, u) is a solution of (2) with

$$P(u) := \frac{\int_{\partial\Omega} u\Phi_1}{\int_{\partial\Omega} \Phi_1^2} = s_n,$$

then (λ, u) is subcritical. Similarly, if $P(u) = s'_n$, then (λ, u) is supercritical. Consequently, there exist two sequences of solutions of (2), *$\{(\lambda_n, u_n)\}$ and $\{(\lambda'_n, u'_n)\}$, converging to $(\sigma_1, 0)$ as $n \to \infty$, one of them subcritical, $\lambda_n < \sigma_1$, and the other supercritical, $\lambda'_n > \sigma_1$.*

(ii) There is a sequence converging to zero of turning points $\{(\lambda^*_n, u^*_n)\}$ such that

$$\lambda^*_n \to \sigma_1, \qquad \|u^*_n\|_{L^\infty(\partial\Omega)} \to 0, \qquad \text{as } n \to \infty.$$

In fact, we can always choose two subsequences of turning points, one of them subcritical, $\lambda^*_{2n+1} < \sigma_1$, and the other supercritical, $\lambda^*_{2n} > \sigma_1$.

(iii) There is a sequence converging to zero of resonant solutions, i.e., there are infinitely many solutions $\{(\sigma_1, \widetilde{u}_n)\}$ *of* (2) *with* $\|\widetilde{u}_n\|_{L^\infty(\partial\Omega)} \to 0$.

The behavior of positive solutions to (2) bifurcating from $(\sigma_1, 0)$ described in Theorems 7.3 and 7.4 is similar to that of the solutions bifurcating from (σ_1, ∞) for the sublinear problem (cf. [9] for details).

The complex nature of the nonlinearity in (122) makes an exhaustive analysis of the global bifurcation diagram outside the scope of this work.

In [25] the author considers the case $\alpha = 1$, $\beta = 1$. He assumes either $N = 1$ or Ω to be a ball, and the nonlinearity to be bounded by a constant small enough. He obtains what he calls an oscillatory bifurcation. We refer the reader to [20], [21] for related problems with nonlinear boundary conditions.

7.18. Subcritical, supercritical and resonant solutions near zero

In this section we give sufficient conditions for the existence of a branch of solutions to (2) bifurcating from zero which is neither *subcritical* $(\lambda < \sigma_1)$, nor *supercritical*, $(\lambda < \sigma_1)$. From this, we conclude the existence of infinitely many *turning points* (see Definition 1.1) and an infinite number of solutions for the resonant problem, i.e., for $\lambda = \sigma_1$. This is achieved in Theorem 7.4

At this step, we analyze when the parameter may cross the first Steklov eigenvalue. To do that, we look at the asymptotic growth rate of the nonlinear term

$$\underline{\mathbf{G}}_{0^+} := \int_{\partial\Omega} \liminf_{s\to 0} \frac{sg(\cdot, s)}{|s|^{1+\alpha}} \, \Phi_1^{1+\alpha} \tag{125}$$

for $\alpha > 1$. Changing lim inf by lim sup, we define the number $\overline{\mathbf{G}}_{0^+}$. If $\underline{\mathbf{G}}_{0^+} > 0$, then $\mathcal{C}^+$ is subcritical, and if $\overline{\mathbf{G}}_{0^+} < 0$, then $\mathcal{C}^+$ is supercritical in a neighborhood of $(\sigma_1, 0)$. See [8, Theorems 3.4and 3.5] for the bifurcation from infinity case. In this Section we consider nonlinearities for which

$$\underline{\mathbf{G}}_{0^+} < 0 < \overline{\mathbf{G}}_{0^+}.$$

We shall argue as in [9] for the bifurcation from infinity case. To determine whether a sequence of solutions (λ_n, u_n) is subcritical or supercritical, one must check the sign of

$$\liminf_{n\to\infty} G(u_n) \qquad \text{and} \quad \limsup_{n\to\infty} G(u_n), \tag{126}$$

where G is defined by (123). This is done in Lemma 7.7.

In Proposition 7.6, it is proved that when g is such that

$$|g(x, s)| = O\left(|s|^\alpha\right) \text{ as } |s| \to 0 \text{ for some } \alpha > 1,$$

then the solutions in $\mathcal{C}^\pm$ can be described as

$$u_n = s_n\Phi_1 + w_n, \qquad \text{where} \quad \int_{\partial\Omega} w_n\Phi_1 = 0 \quad \text{and} \qquad w_n = O(|s_n|^\alpha) \text{ as } n \to 0.$$

We unveil the signs of the expressions in (126) by just looking at the signs of the expressions in (126) at $\lambda_n = \sigma_1$ and $u_n = s_n\Phi_1$. This is achieved in Lemma 7.8.

For this we first consider a family of linear Steklov problems with a variable nonhomogeneous term at the boundary h depending on the parameter λ

$$\left\{ \begin{array}{rcll} -\Delta u + u &=& 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} &=& \lambda u + h(x), & \text{on } \partial\Omega, \end{array} \right. \tag{127}$$

where $h \in L^r(\partial\Omega)$, $r > N-1$ and $\lambda \in (-\infty, \sigma_2)$.

We use the decomposition

$$L^r(\partial\Omega) = \text{span}[\Phi_1] \oplus \text{span}[\Phi_1]^\perp, \quad \text{where} \quad \text{span}[\Phi_1]^\perp := \left\{ u \in L^r(\partial\Omega) : \int_{\partial\Omega} u\,\Phi_1 = 0 \right\}.$$

For $h \in L^r(\partial\Omega)$, with $r > N-1$, we write

$$h = a_1\Phi_1 + h_1, \quad \text{with } a_1 = \frac{\int_{\partial\Omega} h\,\Phi_1}{\int_{\partial\Omega} {\Phi_1}^2}, \quad \int_{\partial\Omega} h_1\Phi_1 = 0. \tag{128}$$

For $\lambda \neq \sigma_1$ the solution $u = u(\lambda)$ of (127) has a unique decomposition

$$u = \frac{a_1}{\sigma_1 - \lambda}\,\Phi_1 + w, \quad \text{where} \quad \int_{\partial\Omega} w\Phi_1 = 0, \tag{129}$$

and $w = w(\lambda) \in \text{span}[\Phi_1]^\perp$ solves the problem

$$\left\{ \begin{array}{rcll} -\Delta w + w &=& 0, & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} &=& \lambda w + h_1(x), & \text{on } \partial\Omega. \end{array} \right. \tag{130}$$

Note that in (130) $w(\lambda) \in \text{span}[\Phi_1]^\perp$ is also well defined for $\lambda = \sigma_1$. Moreover, we have:

Lemma 7.5. *For each compact set $K \subset (-\infty, \sigma_2) \subset \mathbb{R}$ there exists a constant $C = C(K)$, independent of λ, such that*

$$\|w(\lambda)\|_{L^\infty(\partial\Omega)} \leq C\|h_1\|_{L^r(\partial\Omega)} \quad \textit{for any} \quad \lambda \in K,$$

where $w \in$ span$[\Phi_1]^\perp$ is the solution of (130) *and $h_1 \in$ span$[\Phi_1]^\perp$ is defined in* (128).

Proof. See Lemma 3.1 of [9]. ☑

Now we turn our attention to the nonlinear problem (2). Recall that for solutions (λ, u) close to the bifurcation point $(\sigma_1, 0)$ we have

$$u = s\Phi_1 + w, \qquad \text{where } w = o(s), \quad w \in \text{span}[\Phi_1]^\perp \qquad \text{as } s \to 0. \tag{131}$$

We define

$$P(u) := \frac{\int_{\partial\Omega} u\,\Phi_1}{\int_{\partial\Omega} \Phi_1^2}. \tag{132}$$

Next, we give sufficient conditions on the nonlinear term g in (2), for $w = O(|s|^\alpha)$ as $s \to 0$ (see (131)). We restrict ourselves below to the branch of positive solutions; a completely analogous result holds for the branch of negative solutions. The following Proposition is essentially Proposition 3.2 in [9] rewritten for $s \to 0$. See [13, Proposition 2.2] for a proof.

Proposition 7.6. *Assume g satisfies hypotheses* (H5), (H6) *and* (H7).

Then, there exists an open set $\mathcal{O} \subset \mathbb{R} \times C(\bar{\Omega})$ of the form $\mathcal{O} = \{(\lambda, u) : |\lambda - \sigma_1| < \delta_0, \|u\|_{L^\infty(\Omega)} < s_0\}$ for some δ_0 and s_0, such that

(i) *There exists a constant C_1 independent of λ such that, if $(\lambda, u) \in \mathcal{C}^+ \cap \mathcal{O}$ and $(\lambda, u) \neq (\sigma_1, 0)$, then $u = s\Phi_1 + w$, where $s > 0$, $w \in span[\Phi_1]^\perp$ and*

$$\|w\|_{L^\infty(\partial\Omega)} \leq C_1 \|G_1\|_{L^r(\partial\Omega)} \, |s|^\alpha, \qquad as\ |s| \to 0;$$

(ii) *there exists a constant $S_0 > 0$ such that for all $|s| \leq S_0$ there exists $(\lambda, u) \in \mathcal{C}^+ \cap \mathcal{O}$ satisfying $u = s\Phi_1 + w$, with $w \in span[\Phi_1]^\perp$.*

(iii) *Moreover, for any $(\lambda, u) \in \mathcal{C}^+ \cap \mathcal{O}$, $u = s\Phi_1 + w$, with $w \in span[\Phi_1]^\perp$,*

$$|\sigma_1 - \lambda| \leq C_2 |s|^{\alpha - 1}, \qquad as\ |s| \to 0,$$

with C_2 independent of λ; in fact,

$$C_2 = \frac{2\|G_1\|_{L^1(\partial\Omega)}}{\int_{\partial\Omega} \Phi_1^2}.$$

Our next Lemma is essentially Lemma 3.1 in [8] rewritten for $s \to 0$. It allows us to estimate $\sigma_1 - \lambda_n$ as λ_n converges σ_1. See [13, Lemma 2.2] for a proof.

Lemma 7.7. *Assume the nonlinearity g satisfies hypotheses* (H5), (H6) *and* (H7).

Let (λ_n, u_n) be a sequence of solutions of (2) with $\lambda_n \to \sigma_1$ and $\|u_n\|_{L^\infty(\partial\Omega)} \to 0$. If $u_n > 0$, then

$$\begin{aligned} \frac{\underline{\mathbf{G}}_{0^+}}{\int_{\partial\Omega} {\Phi_1}^2} &\leq \frac{1}{\int_{\partial\Omega} {\Phi_1}^2} \liminf_{n\to\infty} G(u_n) && (133) \\ &\leq \liminf_{n\to\infty} \frac{\sigma_1 - \lambda_n}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \leq \limsup_{n\to\infty} \frac{\sigma_1 - \lambda_n}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}} && (134) \\ &\leq \frac{1}{\int_{\partial\Omega} {\Phi_1}^2} \limsup_{n\to\infty} G(u_n) \leq \frac{\overline{\mathbf{G}}_{0^+}}{\int_{\partial\Omega} {\Phi_1}^2}. \end{aligned}$$

A similar statement is obtained for the case $u_n < 0$, just changing $\underline{\mathbf{G}}_{0^+}$ by $\underline{\mathbf{G}}_{0^-}$ and $\overline{\mathbf{G}}_{0^+}$ by $\overline{\mathbf{G}}_{0^-}$.

Let $\{s_n\}$ and $\{s'_n\}$ satisfy

$$-\infty < \lim_{n\to+\infty} G(s'_n\Phi_1) < 0 < \lim_{n\to+\infty} G(s_n\Phi_1) < \infty. \tag{135}$$

In order to prove Theorem 7.4, we show that the signs in (126) can be deduced from those of (135). This is stated in the following result. See [13, Lemma 2.4] for a proof.

Lemma 7.8. *Assume that g satisfies hypotheses* (H5)*,* (H6) *and* (H7)*.*

If $(\lambda_n, s_n) \to (\sigma_1, 0)$ and there exists a constant C such that $\|w_n\|_{L^\infty(\partial\Omega)} \le C|s_n|^\alpha$ for all $n \to 0$, then

$$\liminf_{n\to+\infty} G(s_n\Phi_1 + w_n) \ge \liminf_{n\to+\infty} G(s_n\Phi_1),$$

where G is given by (123)*. Similarly,*

$$\limsup_{n\to+\infty} G(s_n\Phi_1 + w_n) \le \limsup_{n\to+\infty} G(s_n\Phi_1).$$

Remark 7.9. With respect to the stability of the solutions, let us recall Remark 4.1.

7.19. Two examples

Resonant solutions for an oscillatory nonlinearity

Let us consider the oscillatory nonlinearity given by equation (122). In Theorem 7.1 it is proved that if $\alpha > 1$, for any $\beta \in \mathbb{R}$, and $C \in \mathbb{R}$, there is an unbounded branch of positive solutions. Assume from now that $\beta < 0$.

Taking $|C| \le 1$, it is not difficult to see that

$$u_k(x) := [\operatorname{asin}(-C) + k\pi]^{1/\beta}\Phi_1(x), \qquad k \ge 0,$$

defines a sequence of resonant solutions to (2) such that $u_k(x) \to 0$ as $k \to \infty$.

A one dimensional example

Now we consider the onedimensional version of (2).

Fix now

$$g(s) = s^\alpha \ \sin(s^\beta) \qquad \text{for any } \alpha > 1,\ \beta < 0.$$

From definition (125) we can write

$$\begin{aligned}
\underline{\mathbf{G}}_{0^+} &:= \int_{\partial\Omega} \liminf_{s\to 0^+} \frac{sg(s)}{|s|^{1+\alpha}}\ \Phi^{1+\alpha} = \int_{\partial\Omega} \liminf_{s\to 0^+}\ \sin(s^\beta)\ \Phi^{1+\alpha} = -\int_{\partial\Omega} \Phi^{1+\alpha} < 0,\\
\overline{\mathbf{G}}_{0^+} &:= \int_{\partial\Omega} \limsup_{s\to 0^+} \frac{sg(s)}{|s|^{1+\alpha}}\ \Phi^{1+\alpha} = \int_{\partial\Omega} \limsup_{s\to 0^+}\ \sin(s^\beta)\ \Phi^{1+\alpha} = \int_{\partial\Omega} \Phi^{1+\alpha} > 0,
\end{aligned}$$

and then $\underline{\mathbf{G}}_{0^+} < 0 < \overline{\mathbf{G}}_{0^+}$.

Moreover, by looking in (37) at the values of $s \in \mathbb{R}$ such that $\lambda(s) = \sigma_1$, it is easy to check that (σ_1, u_k) is a solution for any $k \in \mathbb{Z}$, where

$$u_k(x) := \frac{(k\pi)^{1/\beta}}{e+1}(e^x + e^{1-x}),$$

i.e., there is a sequence of solutions of the resonant problem converging to zero (see Fig. 11).

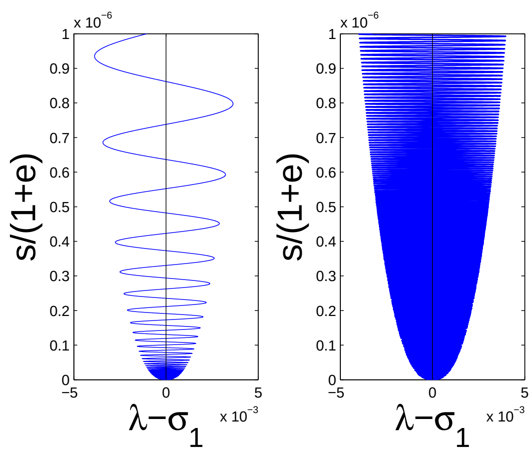


Figure 11. $\alpha = 1.4$, $\beta = -0.3$ and $\beta = -0.5$.

Moreover, computing in (37) the local maxima and minima of $\lambda(s)$ it is not difficult to check that (λ_k^*, u_k^*) is a sequence of turning points converging to zero, where

$$\lambda_k^* := \sigma_1 - t_k^{(\alpha-1)/\beta} \sin(t_k), \qquad u_k^*(x) := t_k^{1/\beta}(e^x + e^{1-x}),$$

and where t_k is such that

$$\tan(t_k) = -\frac{\beta}{\alpha - 1} t_k, \qquad t_k \in [-\pi/2 + k\pi, \pi/2 + k\pi],$$

with $t_k \to \infty$ and $t_k^{1/\beta} \to 0$ as $k \to \infty$, thanks to $\beta < 0$.

Let us observe that the bifurcation from zero phenomena occurs whenever $\alpha > 1$ for any β, and that whenever $\alpha + \beta < 1$ the number of oscillations grows up quicker than the number of oscillations of multiples of the eigenfunction and can not be controlled (let us compare Fig. 11 left and right).

8. *The tangential variation of a localized flux-type eigenvalue problem*

In this Section we present (without proof) a formula for the derivative of the principal eigenvalue $\sigma = \sigma_1(\Gamma)$ to the localized Steklov problem

$$\begin{cases} -\Delta u + q(x)u &= 0, \qquad x \in \Omega, \\ \dfrac{\partial u}{\partial n} &= \sigma \chi_\Gamma(x) u, \qquad x \in \partial\Omega, \end{cases} \tag{136}$$

where $\Omega \subset \mathbb{R}^N$ is a class C^3 bounded domain with boundary $\partial\Omega$ and outer unit normal field $n = n(x)$, $\Gamma \subset \partial\Omega$ is a smooth subdomain of $\partial\Omega$ and χ_Γ is its characteristic function relative to $\partial\Omega$, ($\chi_\Gamma = 1$ if $x \in \Gamma$, $\chi_\Gamma = 0$ for $x \in \partial\Omega \setminus \Gamma$). We obtain an explicit formula for the derivative of $\sigma_1(\Gamma)$ with respect to Γ. The lack of regularity up to the boundary of the first derivative of the principal eigenfunctions is a further intrinsic feature of the problem. Therefore, the whole analysis must be done in the weak sense of $H^1(\Omega)$. The study is of interest in mathematical models in morphogenesis.

Throughout this Section, it will be always assumed that Γ is a subdomain (an open connected set) so that $\overline{\Gamma} = \Gamma \cup \partial\Gamma$ defines a class C^3 closed submanifold of $\partial\Omega$ with boundary $\partial\Gamma$. We will refer to this requirement of the flux region Γ in the sequel by saying that Γ is a *smooth subdomain* of $\partial\Omega$. In addition, the potential term q will be supposed C^1 up to the boundary, i.e., $q \in C^1(\overline{\Omega})$.

The main objective of this Section is to show that the principal eigenvalue to problem (136) varies in a smooth way when the flow region Γ is "tangentially" deformed according to a broad class of regular perturbations (cf. [29, Section 3] for precise definitions). Furthermore, an explicit formula for the variation of such eigenvalue with respect to Γ is obtained (Theorem 8.1). Accordingly, the perturbation problem addressed here falls in the realm of "variation of domains", a field with long tradition in the theory of linear and nonlinear eigenvalue problems (cf. the specific monography [22] on the subject, [33] and [27] together with its references).

Problem (136) can be observed as a Steklov problem where the flux through the boundary is restricted, by means of the weight function χ_Γ, to a specific zone Γ of $\partial\Omega$ (cf. [7] and [20] for related Steklov problems). Our main interest will be focused on principal eigenvalues. By a *principal eigenvalue* to (136) it is understood an eigenvalue σ with a positive associated eigenfunction Φ. It can be shown that (136) admits an eigenvalue exhibiting that property if and only if the first eigenvalue of the mixed problem

$$\begin{cases} -\Delta\phi + q\phi &= \nu\phi, \qquad x \in \Omega, \\ \phi &= 0, \qquad x \in \Gamma, \\ \dfrac{\partial\phi}{\partial n} &= 0, \qquad x \in \partial\Omega \setminus \overline{\Gamma}, \end{cases} \tag{137}$$

is positive. Moreover, there only exists a unique principal eigenvalue σ_1.

The principal eigenvalue plays a crucial role when one deals with natural perturbations of (136), and the interest is put in positive solutions. Specifically, consider the problem

$$\begin{cases} -\Delta u + q(x)u &= f(x,u), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} &= \chi_\Gamma(x)(\sigma u + g(x,u)), & x \in \partial\Omega, \end{cases} \tag{138}$$

where $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ define certain volumetric and surface reaction terms, respectively. Assume that $f(x,u) = uf_1(x,u)$, $g(x,u) = ug_1(x,u)$ with both f_1 and g_1 continuously differentiable and satisfying $f_1(x,0) = g_1(x,0) = 0$ in Ω. Then, problem (138) can be regarded as a model for a chemical reactor Ω where the species u is consumed in a rate $-q+f_1$ meanwhile it is pumped into the reactor with a flux-intensity σ through the window Γ in the boundary $\partial\Omega$ (cf. [18] for related ideas). In fact, a positive solution u to (138) –if such a solution exists– provides the equilibrium regime of production for such a substance u. In other words, a positive stationary solution to the reaction-diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + q(x)u &= f(x,u), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} &= \chi_\Gamma(x)(\sigma u + g(x,u)), & x \in \partial\Omega. \end{cases} \tag{139}$$

Suppose now that both f_1 and g_1 are decreasing. A simple computation reveals that a necessary condition for the existence of such a positive solution is that the intensity σ be greater than σ_1. Furthermore, $\sigma > \sigma_1$ turns out to be also a sufficient condition for the existence of a unique positive equilibrium, provided $f_1(x,u) \to -\infty$, $g_1(x,u) \to -\infty$ as $u \to \infty$ (cf. [20] for precise details together with further configurations for the reaction terms f and g). This means that the system requires a large enough flux intensity σ through the "localized zone" Γ to sustain a stable regime. The critical value of σ is just provided by σ_1. On the other hand, $\sigma = \sigma_1$ constitutes a bifurcation value, either from zero or infinity, for positive solutions of (138) if suitable structure conditions are satisfied by the nonlinearities f and g (cf. [7, 8] and complementary multiplicity results in [9]).

In [16] authors presented a reaction-diffusion model for patterning of limb cartilage development, a paramount problem in embryology ([28]). They considered a growing domain modeling the limb bud (the reactor Ω), and developed a numerical scheme that incorporated the interactions between two distinguished reactants u_1, u_2 located in very specific zones Γ_1, Γ_2 of the boundary $\partial\Omega$. The relevance of such substances u_i (called *morphogens*) and the prominent role of the flux regions Γ_i has been largely supported by a strong experimental evidence ([32], [34]). Experiments also suggests that the pattern-formation seems to be driven by the mutual regulation of the fluxes of u_i through the zones Γ_i.

Inspired in [16], the present section analyzes the phenomenology of the flux zones from an alternative point of view. Since σ_1 measures the threshold value of σ in order that (138) exhibits a positive solution, a special emphasis should be put on how does σ_1 varies with Γ. Therefore, the "size" of the region $\Gamma \subset \partial\Omega$ will be regarded here as a parameter in the sense that the whole of Γ will be subject to tangential deformations. Our main purpose will be then to study the corresponding variations of σ_1, as direct response to such perturbation.

Another key feature of problem (136) is the lack of regularity exhibited by the eigenfunctions associated to the principal eigenvalue σ_1. In fact, such eigenfunctions fails to be of class C^1 up to the boundary [29, Section 2, Theorem 2.1]). This singular behavior is caused by the discontinuity of the coefficient χ_Γ through the interphase $\partial\Gamma$ (the boundary of $\partial\Gamma$ in $\partial\Omega$). As a direct consequence of this fact, the full analysis of existence of a principal eigenvalue to (136), and its properties of continuity and differentiability with respect Γ, must be necessarily performed in the "weak" framework of $H^1(\Omega)$.

8.20. The first variation of σ_1 on smooth domains

The objective of this section is showing a formula for the derivative of the principal eigenvalue $\sigma = \sigma_1(t)$ to problem

$$\begin{cases} -\Delta v + q(y)v &= 0, & y \in \Omega, \\ \dfrac{\partial v}{\partial n} &= \sigma\, \chi_{\Gamma_t}(y)v, & y \in \partial\Omega. \end{cases} \tag{140}$$

We introduce the notion of tangential deformation of the flux region $\Gamma \subset \partial\Omega$.

We are considering a class C^2 vector field $V : \partial\Omega \to \mathbb{R}^N$ which is tangent to $\partial\Omega$ at every point. Recall that $\Omega \subset \mathbb{R}^N$ is assumed to be a class C^3 bounded domain. Hence, the field V can be extended as a smooth field on the whole $\mathbb{R}^N$ in such a way that $V \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$.

Associated to the field V we set $h : \mathbb{R} \times \partial\Omega \to \partial\Omega$ the flow generated by V. Namely, for $x_0 \in \partial\Omega$, $x(t) = h(t, x_0)$ stands for the solution to the initial value problem

$$\begin{cases} \dfrac{dx}{dt} &= V(x), \\ x(0) &= x_0. \end{cases}$$

Γ_t is designating the perturbation at time t of a smooth subdomain $\Gamma \subset \partial\Omega$, through the flow $h = h(t, x)$ of a class C^2 tangential field V on $\partial\Omega$.

The formula for the derivative is obtained in next result (see the following Theorem). We do not include its proof, which is beyond the scope of this work (cf. [29] for detailed proofs).

Theorem 8.1. *Let $\overline{\Gamma} = \Gamma \cup \partial\Gamma \subset \partial\Omega$, $\overline{\Gamma} \neq \partial\Omega$, be a smooth and connected $N-1$-dimensional manifold with boundary $\partial\Gamma$, while $V : \partial\Omega \to \mathbb{R}^N$ is a smooth tangent vector field to $\partial\Omega$ with associated flow $h : \mathbb{R} \times \partial\Omega \to \partial\Omega$. Setting*

$$\Gamma_t = \{y = h(t, x) : x \in \Gamma\},$$

consider the eigenvalue problem (140) *and assume that* $\nu_1(\Gamma) > 0$.

Assume in addition that Ω is C^∞, $q \in C^\infty(\overline{\Omega})$ and that none of the Neumann eigenvalues of $-\Delta + q$ in Ω vanishes. Then, the derivative of the principal eigenvalue $\sigma_1(t)$ to (140) *is given by the expression*

$$\left.\frac{d\sigma_1}{dt}\right|_{t=0} = -\sigma_1(0) \int_{\partial\Gamma} \Phi_1(0)^2 \langle V, n_{\partial\Gamma}\rangle\, d\Lambda_{\partial\Gamma}, \tag{141}$$

where $n_{\partial\Gamma}$ stands for the outer unit normal field to $\partial\Gamma$ relative to Γ, $d\Lambda_{\partial\Gamma}$ is the volume element of $\partial\Gamma$ and $\Phi_1(0)$ stands for the normalized positive eigenfunction associated to $\sigma_1(0)$.

Bibliographical Notes

Section 2 on bifurcation from infinity is contained in [7]. Specifically, Proposition 2.3 on Hölder regularity of the solutions is proved in [7, Proposition 2.3]. Theorem 2.7 on existence of bounded solutions is proved in [7, Theorem 2.7]. Theorem 2.10 on bifurcation from infinity is proved in [7, Theorem 3.3]. Theorem 2.11 on bifurcation from infinity from a simple eigenvalue, is proved in [7, Theorem 3.4]. Lemma 2.13, to determine a rate of convergence of λ to σ_1, is proved in [8, Lemma 3.1]. It can also be used to determine whether a sequence of solutions lies at one side or another of σ_1. Theorem 2.14 on bifurcation from the first eigenvalue is proved in [7, Theorem 4.3]. Finally, Theorem 2.16 on bifurcation from higher eigenvalues is [7, Theorem 4.5].

Section 3 on the Anti-Maximum Principle and a uniform Anti-Maximum Principle is essentially contained in [7, Sections 5 and 6] and in [8, Appendix] respectively. Theorem 3.1 and Theorem 3.4 are [7, Theorem 6.1] and in [8, Theorem A.3] respectively

Section 4 on the stability analysis of the solutions is essentially contained in [7, Section 7] and in [8]. Theorem 4.5 on the stability of the solutions in the bifurcated from infinity branch is proved in [8, Theorem 3.4]. In [7] a preliminary result of Theorem 4.5 was proved in Proposition 7.1. Theorem 4.6 on the instability of the solutions in the bifurcated from infinity branch is proved in [8, Theorem 3.5]. A preliminary result of Theorem 4.6 was obtained in [7, Proposition 7.3].

Section 5 on turning points and the resonant case is mostly developed in [9]. Theorem 5.1 on Landesman–Lazer type conditions is proved in Theorem 5.1 of [7]. Proposition 5.4 is proved in [9, Proposition 3.2]. Lemma 5.5 is proved in [9, Lemma 3.3], and Theorem 5.6 on the existence of infinitely many resonant solutions and infinitely many turning points is proved in [9, Theorem 3.4].

Section 6 on stability switches is developed in [11].

Section 7 on bifurcation from the trivial solution set is essentially written in [12]. Theorem 7.1 is [7, Theorem] , and Theorem 7.3 and Theorem 7.4 are [12, Theorem 1.2 and Theorem 1.3], respectively.

Section 8 on the derivative of a localized Steklov eigenvalue with respect to tangential variations of the subset of the boundary is entirely contained in [29].

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